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About dual categories and representation theorems for some categories of lattices

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Logica matematica. — *About dual categories and representation theorems for some categories of lattices.* Nota di ALEXANDRU BREZULEANU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si ottengono risultati sulle algebre di Lukasiewicz, mediante teoremi di rappresentazione che forniscono una relazione di dualità fra categorie di natura algebrica (insiemi, gruppi, anelli, reticoli, ecc.) e certe categorie topologiche.

The most important structures of the mathematics, the topological (i.e. continuous) and algebraic ones are dual, i.e. categories of algebraic nature (sets, groups, rings, lattices, etc.) have as duals either categories of some topological spaces or topological spaces with some sheaves [9].

The Note [1] and the present Note prove this result. (We do not know whether and how the notion of Grothendieck topology would explain this duality).

We consider the following categories:

Ld (0, 1)-distributive lattices with a first and a last element; L*rl*-linear residuated lattices [7], [8] named L-algebras in [3]; $\mathrm{L}uk_\theta$ — θ valent Lukasiewicz algebras [6]; $\mathbf{P}_\theta$ — θ-valent Post algebras [2]; **B**-booleian algebras. A lattice is called local if it has a unique maximal ideal (or filter).

We shall give a representation theorem valid in Ld (0, 1) which has as particular cases the known representation theorems in L*rl*, **B**, $\mathrm{L}uk_\theta$, $\mathbf{P}_\theta$. That theorem shows that the models of Ld (0, 1), $\cdots$, **B** are the local lattices. Hence a central problem is to know the local objects of the category. They are more interesting if the respective category has as dual a category of affine schemes of lattices since the stalks are local lattices and they transmit their properties to the whole category.

1. Let $\mathrm{L} \in ob\, \mathrm{Ld}(0, 1)$ and Specm L be the set of the maximal ideals of L. Let $\mathfrak{p}$ be a prime ideal in L and $\mathfrak{f} = \mathrm{L} - \mathfrak{p}$. The localisation $\mathrm{L}_\mathfrak{p}$ of L in $\mathfrak{p}$ coincides with the factor lattice $\mathrm{L}/\mathfrak{f}$ [1]; $\mathrm{L}_\mathfrak{p}$ has a unique maximal ideal.

1.1. LEMMA. *The homomorphism* $i : \mathrm{L} \to \prod_{\mathfrak{p} \in \mathrm{Specm\,L}} \mathrm{L}_\mathfrak{p}$, *given by the canonical maps* $i_\mathfrak{p} : \mathrm{L} \to \mathrm{L}_\mathfrak{p}$, *is injective.*

Proof. Let $a, b \in \mathrm{L}$ with $i_\mathfrak{p}(a) = i_\mathfrak{p}(b)$ for any $\mathfrak{p} \in$ Specm L. Hence for any $\mathfrak{p}$, there is $c_\mathfrak{p} \notin \mathfrak{p}$ with $a \wedge c_\mathfrak{p} = b \wedge c_\mathfrak{p}$. The ideal $\mathfrak{a}$ generated by $\mathrm{I} = \{c_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Specm\,L}\}$ coincides with L (if $\mathfrak{a} \neq \mathrm{L}$, there is $\mathfrak{q} \in$ Specm L with $\mathfrak{a} \subset \mathfrak{q}$, hence $c_\mathfrak{q} \in \mathfrak{q}$). Then there are $c_1, \ldots, c_n \in \mathrm{I}$ with $c_1 \vee \cdots \vee c_n = 1$. Hence $a = \bigvee_1^n (a \wedge c_i) = \bigvee_1^n (b \wedge c_i) = b$.

(*) Nella seduta dell'11 novembre 1972.

1.2. REMARKS. i) For $\mathrm{L}rl$, $\mathbf{B}$ and $\mathrm{L}uk_\theta$, $\mathbf{P}_\theta$ (with θ-ideals and θ-filters instead of ideals, filters) they can prove 1.1 by the fact that the intersection of the minimal filters is reduced to $\{1\}$.

ii) In 1.1 we use only the element 1 of L; obviously, we can enounce 1.1 for maximal filters and use $0 \in \mathrm{L}$.

iii) The Lemma 1.1 is a type of representation theorems for different categories of lattices. The problem is if $i_{\mathfrak{p}}: \mathrm{L} \to \mathrm{L}_{\mathfrak{p}}$ (or $i_{\mathfrak{f}}: \mathrm{L} \to \mathrm{L}_{\mathfrak{f}}$, for $\mathfrak{f}$ a filter) is a morphism in the given category; eventually they consider some special ideals or filters, for instance θ-ideals and θ-filters for $\mathrm{L}uk_\theta$ and P_θ.

Lemma 1.1 shows the importance of local objects, since a formula is true iff it is true in the local lattices. For $\mathrm{Ld}(0, 1)$ we do not know a description of the local lattices; but in $\mathrm{L}rl$, $\mathbf{B}$, $\mathrm{L}uk_\theta$, $\mathbf{P}_\theta$ they are well determined.

1.3. Any local booleian algebra is isomorphic to $\mathrm{L}_2 = \{0, 1\}$. Hence, for $\mathrm{L} \in ob\,\mathbf{B}$, the Lemma 1.1 gives the representation theorem of Stone: Any $\mathrm{L} \in \mathrm{ob}\,\mathbf{B}$ is a subdirect product of algebras L_2. We *conjecture* that: *if* $\mathrm{L} \in ob\,\mathrm{Ld}(0, 1)$ *and is a subdirect product of algebras* L_2, *then* $\mathrm{L} \in ob\,\mathbf{B}$.

1.4. The L-algebras are the algebraic correspondent for the special modal logic [5]. The local L-algebras are exactly the chains (= totally ordered sets with a first and a last element). And Lemma 1.1 is the representation theorem for $\mathrm{L}rl$, given independently by A. Monteiro [8], Gr. C. Moisil [7], and later by A. Horn [3]: Any $\mathrm{L} \in ob\,\mathrm{L}rl$ is a subdirect product of chains. We think the following converse of this is true:

CONJECTURE. *Let* $\mathrm{L} \in ob\,\mathrm{Ld}(0, 1)$. *If* L' *is a direct product of chains,* L *is a subdirect product in* L' *and* L *and* L' *have the same stalks in Spec* L, *then* $\mathrm{L} \in ob\,\mathrm{L}rl$.

1.5. Let J be a set of ordinal type θ. We need the functors $\mathrm{L}uk_\theta \xrightarrow{\mathrm{C}} \mathbf{B} \xrightarrow{\mathrm{D}} \mathrm{L}uk_\theta$ given by: for $\mathrm{L} \in ob\,\mathrm{L}uk_\theta$, $\mathrm{CL} \overset{\mathrm{def}}{=} \{a \in \mathrm{L} \mid$ there is $x \in \mathrm{L}$ with $a \wedge x = 0$, $a \vee x = 1\}$; for $\mathrm{B} \in ob\,\mathbf{B}$, $\mathrm{DB} \overset{\mathrm{def}}{=} \{f: \mathrm{J} \to \mathrm{B} \mid f$ isotone function$\}$. It is known that: D is right adjoint to C; $\mathrm{L} \to \mathrm{DCL}$ is a monomorphism; $\mathrm{B} \to \mathrm{CDB}$ is an isomorphism [6], [2].

1.5.1. PROPOSITION. *Let* $\mathrm{L} \in ob\,\mathrm{L}uk_\theta$; L *is local if and only if it is a subalgebra of* DL_2. *Let* $\mathrm{L} \in ob\,\mathbf{P}_\theta$; L *is local if and only if* $\mathrm{L} = \mathrm{DL}_2$.

Proof. Let L be a local algebra, i.e. Max L {= the set of the prime θ-ideals of L} has a unique element. From Max L = Spec CL, it results that $\mathrm{CL} = \mathrm{L}_2$. Since $\mathrm{L} \to \mathrm{DCL}$ is a monomorphism it follows that $\mathrm{L} \subset \mathrm{DL}_2$. Now let $\mathrm{L} \subset \mathrm{DL}_2$; then $\mathrm{CL} \subset \mathrm{CDL}_2 = \mathrm{L}_2$, hence L is local, since Max L = Spec CL.

For $\mathrm{L} \in ob\,\mathrm{P}_\theta$, we have $\mathrm{L} \simeq \mathrm{DCL}$ and this proves the last statement.

This proposition and the Lemma 1.1 (in which we consider θ-ideals and θ-filters) give in a more precise form the representation theorem for Luk_θ and $\mathbf{P}_\theta$ [6]:

Any $L \in ob\, Luk_\theta$ (resp. *any* $L \in ob\, \mathbf{P}_\theta$) *is a subdirect product in a direct product* L' *of subalgebras of* DL_2 (resp. *of algebras* DL_2) L *and* L' *having the same stalks in* Max L.

I suppose the converse of it is also true, for $L \in ob\, Ld(0, 1)$. Particularly, I *conjecture* that: *if* $L \in ob\, Luk_\theta$ *and for any* $\mathfrak{m} \in \text{Max}\, L$, $L/\mathfrak{m} = DL_2$, *then* $L \in ob\, \mathbf{P}_\theta$.

1.5.2. It is known that for $L \in ob\, Ld(0, 1)$ and L infinite, card $L \leq$ card Spec L [4].

If $L \in ob\, Luk_\theta$ and L is infinite, this is not true in the form card $L \leq$ card Max L, for instance taking $L = DL_2$. However, *if* $L \in ob\, Luk_\theta CL$ *is infinite and* θ *is finite, then card* $L \leq$ card Max L. Indeed, card $L \leq$ card DCL $=$ $=$ card CL for θ finite and CL infinite, and also card CL $\leq$ card Spec CL by [4], and Spec CL $=$ Max L.

2. By Stone–Nerode it is known that the dual of $Ld(0, 1)$ is the category of Stone spaces with compact continuous maps. By [2], the dual of Luk_θ is the category of Lukasiewiecz spaces with compact continuous maps commuting with the Lukasiewiecz endomorphisms. It seems that an analogous duality holds for Lrl. The dual of $\mathbf{B}$ is the category of booleian spaces with continuous maps. The inconvenient of the first dualities is that they do not take into account the local objects, i.e. the models. As for $\mathbf{B}$ (and also for $\mathbf{P}_\theta$ which is equivalent to $\mathbf{B}$), the local objects are all isomorphic to L_2 (resp. DL_2), and so the topological space of the θ–prime ideals preserves and can give all information.

Another way to obtain dual categories is to consider on the topological spaces some structural sheaves of lattices.

In [1] it is proved that $Ld(0, 1)$ has as dual the category Sal of affine schemes of lattices.

THEOREM. *The dual category of* Lrl *is the full subcategory* $\mathcal{C}$ *of* Sal *consisting of those objects for which the stalks of the structural sheaf are chains.*

Proof. Let $(X, L_X) \in ob\, \mathcal{C}$. It is known that $L = \Gamma(X, L_X) \in ob\, Ld(0, 1)$. We must show that $L \in ob\, Lrl$.

The residuation " : " can be defined in terms of equations (the equalities: $((a : b) \wedge b) \vee a = a$; if $(c \wedge b) \vee a = a$ than $(a : b) \vee c = a : b$, determine uniquely the element $a : b$). Therefore, for $a, b \in L$ it suffices to define the element $a : b$ in every stalk. Since, for $\mathfrak{p} \in \text{Spec}\, L = X$ the element $a : b \in L_\mathfrak{p}$ can be lifted to a neighbourhood U of $\mathfrak{p}$ preserving the above equalities, hence it gives the residuation of a and b in every $L_\mathfrak{q}$ for $\mathfrak{q} \in U$.

In order to obtain $a : b$ in L they glue together the elements so obtained on U's. The residuation is linear since it is linear in every stalk.

If $g:(X, L_X) \to (X', L_{X'})$ is a morphism in $\mathcal{C}$ then $f = \Gamma(X, g): L = \Gamma(X, L_X) \to L' = \Gamma(X', L_{X'})$ is a morphism in Ld (0, 1) and, moreover, it commutes with the residuation, since it does so on stalks (a morphism form Ld (0, 1) between chains is a fortiori isotone).

COROLLARY. *If* L, L′ ∈ *ob* L*rl and* $f: L \to L'$ *is a morphism in* Ld (0, 1) *then it is also in* L*rl*, i.e. *commutes with residuation.*

REMARKS i). Analogously, they can describe the dual of the category L*r* of residuated lattices; but in this case the local objects are not known.

ii) It would be interesting to obtain a similar theorem for Luk_θ using Max L instead of Spec L.

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