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Cubical Polyhedra and Honiotopy. Nota IV

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Topologia. — *Cubical Polyhedra and Homotopy.* Nota IV di WŁODZIMIERZ HOLSZTYŃSKI e JÓZEF BLASS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si continuano tre Note con lo stesso titolo apparse in questi « Rendiconti ». Nella presente Nota IV ed in una successiva Nota V viene costruita una teoria omologica del tipo di Čech basata sullo schema cubico. Più precisamente, si definisce l'omologia cubica di Čech di uno spazio compatto X come il limite inverso delle omologie combinatorie di poliedri approssimativi QX (ved. [1]). Si mostra poi che la teoria omologica così costruita soddisfa agli assiomi di Eilenberg–Steenrod per le teorie omologiche continue definite sulla categoria degli spazi compatti.

INTRODUCTION

In the present paper we construct a cubical Čech homology. The basis of our construction is the functorial assignment to every compact space X of a cubical polyhedron QX having the homotopy type of the space X (see [1]). For the basic definitions used in this paper, the reader is referred to [1] and [2].

Here is an outline of the paper.

Sections 1, 2, 3. Standard definitions of oriented faces of cubical polyhedra and chain complexes of cubical polyhedra are described.

Section 4. The main assertion of the section (4.5) gives us as a corollary the acyclicity of the faces of a cube.

Section 5. Here the cubical carrier theorem (5.5) is proved. The main object of this section is to construct the functor from the contiguous category (see [2]) restricted to finite polyhedra into the category Ch of chain complexes and chain homotopy classes of chain morphisms.

Sections 6, 7. Combinatorial homology of cubical polyhedra is defined and is shown to satisfy Eilenberg–Steenrod's axioms.

Sections 8, 9, 10, 11. The cubical Čech homology of a compact pair (X, X_0) is defined as the inverse limit of the combinatorial homologies of finite polyhedra approximating $Q(X, X_0)$.

We show that the theory of homology constructed in the above manner satisfies the Eilenberg–Steenrod axioms for the continuous homology theory defined on the category of compact spaces.

(*) Nella seduta dell'11 novembre 1972.

§ 1. INCIDENCE COEFFICIENTS OF PERMUTATIONS AND ORIENTATION OF FACES

We will introduce now notation which will lead to definitions of oriented faces of cubical polyhedra and chain complexes of cubical polyhedra. Let B be a finite n–element set. Let $\varphi : \{1, \cdots, n\} \to B$ and $\varphi_1 : \{1, \cdots, n\} \to B$ be two permutations of B. We define an incidence number

$$(1.1) \qquad [\varphi : \varphi_1] = \begin{cases} -1 & \text{when } \varphi \text{ and } \varphi_1 \text{ are of the opposite parity} \\ 1 & \text{when } \varphi \text{ and } \varphi_1 \text{ are of the same parity.} \end{cases}$$

Let $B_0 \subset B$ be a subset of B such that $B \setminus B_0 = \{a\}$ and card $B_0 = k$. Let $\varphi_0 : \{1, \cdots, k\} \to B_0$ and $\varphi : \{1, \cdots, k+1\}$ be permutations of B_0 and B respectively. Let a permutation φ_1 of the set B be given by

$$(1.2) \qquad \varphi_1(i) = \begin{cases} a & \text{if } i = 1 \\ \varphi_0(i-1) & \text{if } 2 \leq i \leq k+1. \end{cases}$$

We define an incidence number $\{\varphi : \varphi_0\}$ as follows:

$$(1.3) \qquad \{\varphi : \varphi_0\} \overset{\text{def}}{=} [\varphi : \varphi_1].$$

We observe, that $\{\varphi : \varphi_0\}$ depends only on the parity classes of φ and φ_0 and that a change of one of the parity classes into the opposite one will change the sign of $\{\varphi : \varphi_0\}$. Let $F_\beta \subset I^A$ be a face of a cube given by $\beta : B \to \{-1, 1\}$. An orientation of a face F_β is defined as follows:

(1.4) DEFINITION. If dim $F_\beta \leq 1$ then an orientation of F_β is a choice of 1, -1.

If dim $F_\beta > 1$ then an orientation of F_β is a choice of a permutation class of the set $A \setminus B$.

We will use $\pi(F_\beta)$ to denote an oriented face.

§ 2. INCIDENCE COEFFICIENT OF FACES

Let $\pi(F_\beta)$ and $\pi'(F_{\beta'})$ be orientations of faces of the cube I^A, where β and β' are defined on B and B' respectively. We now define the incidence coefficient μ of the pair of orientations:

$$(2.1) \qquad \mu = [\pi' : \pi] \neq 0 \quad \text{iff } B' \subset B;\ \text{card } (B \setminus B') = 1 \quad \text{and} \quad \beta | B = \beta'.$$

If the above conditions are satisfied and if $B \setminus B' = \{a\}$ we define:

a) $\mu = \pi' \cdot \pi$ if dim $F_\beta = 0$

b) $\mu = \pi \cdot \{\pi' : \pi_\alpha\}$ if dim $F_\beta = 1$

c) $\mu = \{\pi' : \pi\}$ if dim $F_\beta \geq 2$.

Putting an orientation on a polyhedron $W \subset I^A$ is to assign to each of its faces an orientation. More precisely an orientation of the polyhedron I^A is a function γ from the set of faces of W into the set of their orientations such that $\gamma(F_\beta)$ is an orientation of F_β.

(2.2) LEMMA. *Let γ be an orientation of a cube I^A. Let F_β be a k–dimensional face of I^A and let $F_{\beta''}$ be a $(k-2)$–dimensional face of I^A. Then $\sum_{F_{\beta'}} [\gamma(F_{\beta'}) : \gamma(F_{\beta'})] \cdot [\gamma(F_{\beta'}) : \gamma(F_{\beta''})] = 0$ where $F_{\beta'}$ are $(k-1)$–dimensional faces of I^A.*

Proof. Assume that β and β'' are defined on B and B'' respectively [Note that B might be empty]. In this case we can find $a_1 \neq a_2$ such that $a_1, a_2 \in A \setminus B$ and $B'' = \{a_1, a_2\} \cup B$, $\beta'' | B = \beta$. Then there are only two elements F_{β_i} in the sum given by restriction of β'' to $\{a_i\} \cup B$ for $i = 1, 2$. Hence,

$$\sigma = \sum_{F_{\beta'}} [\gamma(F_\beta) : \gamma(F_{\beta'})] \cdot [\gamma(F_{\beta'}) : \gamma(F_{\beta''})] =$$

$$= [\gamma(F_\beta) : \gamma(F_{\beta_1})] \cdot [\gamma(F_{\beta_1}) : \gamma(F_{\beta''})] + [\gamma(F_\beta) : \gamma(F_{\beta_2})] : [\gamma(F_{\beta''})] .$$

Let us consider σ. It has a constant absolute value independent of the choice of

$$\pi = \gamma(F_\beta) \quad , \quad \pi_1 = \gamma(F_{\beta_1}) \quad , \quad \pi_2 = \gamma(F_{\beta_2}) \quad , \quad \pi'' = \gamma(F_{\beta''}).$$

Therefore, we may assume that the sets $A \setminus B''$, $A \setminus B'$ and $A \setminus B$ satisfy

$$A \setminus B'' \subset \{a_2\} \cup (A \setminus B'') = A \setminus B' \subset \{a_1, a_2\} \cup A \setminus B'' = A \setminus B$$

and that they are ordered so that

$$[\pi_2 : \pi''] = [\pi : \pi_2] = [\pi_1 : \pi''] = 1 .$$

It is easy to see that $[\pi : \pi_1] = -1$ and therefore $\sigma = 0$. This completes the proof of (2.2).

We assume that every face F_β of the cube I^A has been chosen a fixed orientation or (F_β). This assigns to every cubical polyhedron in I^A an orientation. We will use F_β to denote the pair $[F_\beta, \text{ or } (F_\beta)]$.

(2.3) DEFINITION. The incidence coefficient $[F_{\beta'} : F_\beta]$ of the pair of faces of the cube I^A is given by:

$$[F_{\beta'} : F_\beta] \overset{\text{def}}{=} [\text{or } (F_{\beta''}) : \text{or } (F_\beta)].$$

§ 3. CHAIN COMPLEX OF A CUBICAL POLYHEDRON

We define a group of k–dimensional chains $C_k(I^A)$ as follows:

if $k < 0$ then $C_k(I^A) = 0$

if $k \geq 0$ then $C_k(I^A)$ is the free group generated by k–dimensional faces of I^A with the chosen orientation. We now define the boundary operation $\delta_k : C_k(I^A) \to C_{k-1}(I^A)$ as follows:

$$\text{(3.1)} \qquad \begin{aligned} &\delta_k = 0 \quad \text{if} \quad k \leq 0 \\ &\delta_k(F) = \sum_{F_{\beta'}} [F_\beta : F_{\beta'}] \cdot F_{\beta'} \quad \text{if } k > 0 \,. \end{aligned}$$

Notice that only the generators of $C_{k-1}(I^A)$ can occur in the above sum with non-zero coefficients. We have

(3.2) LEMMA. *Let* $F_\beta \in C_k(I^A)$ *and let* $F_{\beta''} \in C_{k-2}(I^A)$ *and* $k \geq 0$. *Then*

$$\sum_{F_{\beta'} \in C_{k-1}(I^A)} [F_\beta : F_{\beta'}]\,[F_{\beta'} : F_\beta] = 0\,.$$

Proof. Apply Lemma (2.2) for $\gamma = \text{or}$. □

From Lemma (3.2) we obtain the familiar result:

(3.3) THEOREM. $\delta_{k-1} \circ \delta_k = 0$. □

Let $V \subset I^A$ be a cubical polyhedron. We define $C_k(V)$ — a group of k–dimensional chains of V as a subgroup of $C_k(I^A)$ generated by (oriented) faces of V. Since $\delta_k(C_k(V)) \subset C_{k-1}(V)$ we have obtained a chain complex $C(V) = \{C_k(V)\,;\,\delta_k\}$. If $V = \varphi C_k(V) = 0$ for every integer k. Let (V, V_0) be a pair of cubical polyhedra in I^A. We define $C_k(V, V_0)$ to be $C_k(V)/C_k(V_0)$ and we call $C_k(V, V_0)$ the group of k–dimensional chains of the pair (V, V_0). We will identify $C_k(V)$ and $C_k(V, \varphi)$. Notice that the commutativity of the diagram

$$\begin{array}{ccc} C_k(V_0) & \longrightarrow & C_k(V) \\ \delta_k \Big\downarrow & & \Big\downarrow \delta_k \\ C_{k-1}(V_0) & \longrightarrow & C_{k-1}(V) \end{array}$$

implies that δ_k induces a homomorphism

$$\delta_k : C_k(V, V_0) \to C_{k-1}(V, V_0).$$

We have thus assigned to a pair of cubical polyhedra (V, V_0) a chain complex

$$C(V, V_0) = \{C_k(V, V_0)\,,\,\delta_k\}\,.$$

We recall that a chain complex is acyclic in dimension n iff the sequence

$$C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \quad \text{is exact.}$$

A chain complex is acyclic iff it is acyclic in all dimensions.

(3.3) REMARK. For a chain complex $C(V)$ we can define an augmentation $\varepsilon : C(V) \to Z$ as follows:

$$\varepsilon(F_\beta) = \text{or}\,(F_\beta) \cdot \prod_{a \in B} \beta(a)\,.$$

It is easy to notice that $\varepsilon \cdot \delta_1 = 0$. If V is connected and non-empty then ε and $-\varepsilon$ are unique homomorphisms of $C_0(V) \to Z$ making the augmented complex acyclic in dimension 0.

§ 4. CYLINDER

Let A be a finite set and let $a \in A$. Let us assume that (V, V_0) is a pair of $(A \setminus \{a\})$ - cubical polyhedra. $(V_0 \subset V \subset I^{A \setminus \{a\}})$ and let $q : A \setminus \{a\} \to A$ be the inclusion.

The pair $I_a(V, V_0) = (f_q^{-1}(V), f_q^{-1}(V_0))$ of A–cubical polyhedra is said to be cylinder (a–cylinder) over (V, V_0).

(4.1) PROPOSITION. *Inclusion* $q : A \setminus \{a\} \to A$ *is a cubical morphism of* $I_a(V, V_0)$ *into* (V, V_0) *for every pair of* $(A \setminus \{a\})$ - *polyhedra* (V, V_0).

Proof of (4.1) is obvious.

(4.2) A pair

$$I_a^{-1}(V, V_0) = F_{(a,-1)} \cap I_a(V, V_0) = (F_{(a,-1)} \cap f_q^{-1}(V), F_{(a,-1)} \cap f_q^{-1}(V_0))$$

is said to be the lower base of the cylinder $I_a(V, V_0)$. Similarly, $I_a^{+1}(V, V_0) = F_{(a,+1)} \cap I_a(V, V_0)$ is called the upper base.

We define the chain homomorphism $C(q) : C(I_a(V, V_0)) \to C(V, V_0)$ as follows

$$(4.3) \qquad C(q)[F_\beta] = \begin{cases} 0 & a \in B \\ \beta(a) \cdot [\mathrm{or}(F_\beta) : \mathrm{or}(f_q(F_\beta))] \cdot f_q(F_\beta) & \end{cases}$$ [1].

Geometrically $C(q)$ is the projection of the faces of I^A onto $I^{A \setminus \{a\}}$ and it decreases the dimension of the faces not lying in the bases. For this reason we define $C(q)$ to be zero on these faces. Let F_β be a face of the polyhedron $V \subset I^{A \setminus \{a\}}$. We define a chain homomorphism

$$i_\varepsilon : C(V, V_0) \to C(I_a(V, V_0))$$

as follows:

$$(4.4) \qquad i_\varepsilon(F_\beta) = \varepsilon \cdot [\mathrm{or}(F_\beta) : \mathrm{or}(F_{\beta_\varepsilon})] \cdot F_{\beta_\varepsilon}$$

where $\beta_\varepsilon : B \cup \{a\} \to \{-1, 1\}$ is given by

$$\beta_\varepsilon(x) = \begin{cases} \beta(x) & x \neq a \\ \varepsilon & x = a . \end{cases}$$

(4.5) THEOREM.

a) $C(q) \circ i_\varepsilon = \mathrm{Id}_{C(V,V_0)}$

b) $i_\varepsilon \circ C(q) \simeq \mathrm{Id}_{C(I_a(V,V_0))}$

for $\varepsilon = \pm 1$ ($\simeq$ *denotes chain homotopy*).

(1) Note that $[\mathrm{or}(F_\beta) : \mathrm{or}(f_q(F_\beta))]$ is defined since $A \setminus (a) \setminus B = A \setminus B$.

Before we prove (4.5) we will state three corollaries.

(4.6) COROLLARY 1. *i_{+1} is chain homotopic to i_{-1}.* □

This can be considered as a combinatorial homotopy axiom.

(4.7) COROLLARY 2. *The chain complex of a cylinder is isomorphic to the chain complex of its bases.* □

(4.8) COROLLARY 3. *The faces of a cube are acyclic.*

Proof (of Corollary 3). The proof will proceed by induction. Let k be the dimension of F_β. When $k = 0$ then F_β is a point and our assertion was proved in (3.3). If $k > 0$ then F_β is a cylinder over a $(k-1)$–dimensional face F'_β. By the induction hypothesis F'_β is acyclic. Hence from (4.7) F_β is acyclic.

Proof (of the Theorem (4.5)). We will prove the theorem for the case $\varepsilon = +1$ only, the proof for $\varepsilon = -1$ is similar. We prove (*a*) first.

$$\begin{aligned}(C(q) \circ i_{+1})(F_\beta) &= C(q)([\text{or }(F_\beta) : \text{or }(F_{\beta_{+1}})] \cdot F_{\beta_{+1}} = \\ &= [\text{or }(F_\beta) : \text{or }(F_{\beta_{+1}})] \cdot C(q)(F_{\beta_{+1}}) = \\ &= [\text{or }(F_\beta) : \text{or }(F_{\beta_{+1}})] \cdot [\text{or }(F_{\beta_{+1}}) : \text{or } f_q(F_{\beta_{+1}})] \cdot f_q(F_{\beta_{+1}}).\end{aligned}$$

Note that pointwise $f_q(F_{\beta_{+1}}) = F_\beta$. Hence:

$$\begin{aligned}(C(q) \circ i_{+1})(F_\beta) &= [\text{or }(F_\beta : \text{or }(F_{\beta_{+1}})] \cdot [\text{or }(F_{\beta_{+1}}) : \text{or }(F_\beta)] \cdot F_\beta = \\ &= ([\text{or }(F_\beta) : \text{or }(F_{\beta_{+1}})])^2 \cdot F_\beta = F_\beta .\end{aligned}$$

We now prove (*b*).

Assume that $F_\beta \subset I^A$ is a face such that $a \notin B$. We define F^a_β to be face of I^A given by $\beta \mid B \setminus \{a\} \to \{-1, 1\}$. We define the chain homomorphism Δ_k as follows:

$$\Delta_k : C_k(I(_a V)) \to (I_a(V))$$

$$\Delta_k(F_\beta) = \begin{cases} 0 & \text{if } a \in B \text{ or } \beta(a) = 1 \\ -[\text{or }(F^a_\beta) : \text{or }(F_\beta)] \cdot F^a_\beta . \end{cases}$$

We show that

$$\delta \circ \Delta + \Delta \circ \delta = i_{+1} \circ C(q) - \text{Id}_{C(I_a(V,V_0))} \tag{4.9}$$

$$i_{+1} \circ C(q)(F_\beta) = \begin{cases} 0 & \text{if } a \in B \\ F_\beta & \text{if } \beta(a) = 1 \\ -[\text{or }(F_\beta) : \text{or }(F_{\beta_{+1}})] \cdot F_{\beta_{+1}} & \text{if } \beta(a) = -1 \end{cases}$$

$$(i_{+1} \circ C(q) - \text{Id}_{C(I_a(V,V_0))})(F_\beta) = \begin{cases} -F_\beta & \text{if } a \in B \\ 0 & \text{if } \beta(a) = 1 \\ -[\text{or }(F_\beta) : \text{or }(F_{\beta_{+1}})] \cdot F_{\beta_{+1}} - F_\beta & \text{if } \beta(a) = -1. \end{cases}$$

i) Assume that $a \in B$.

$$(\delta\Delta)(F_\beta) = 0 .$$

$$(\Delta\delta)(F_\beta) = \Delta([\text{or}(F_\beta) : \text{or}(F_{\beta_{+1}})]) \cdot F_{\beta_{+1}} + \Delta([\text{or}(F_\beta) : \text{or}(F_{\beta_{-1}})]) \cdot F_{\beta_{-1}} +$$

$$+ \Big(\sum_{\substack{F'_\beta \in F_\beta \\ \beta'(a) \neq \pm 1}} [F_\beta : F'_\beta] \cdot F'_\beta \Big)$$

$$= - [\text{or}(F_\beta) : \text{or}(F_{\beta_{-1}})] \cdot [\text{or}(F^a_{\beta_{-1}}) : \text{or}(F_{\beta_{-1}})] \cdot F^a_{\beta_{-1}} .$$

Notice that pointwise $F_\beta = F^a_{\beta_{-1}}$. Hence

$$(\Delta\delta)(F_\beta) = - F_\beta .$$

Thus (4.9) holds in this case.

ii) Assume that $\beta(a) = 1$.

$$\delta\Delta(F_\beta) = 0$$

$$(\Delta\delta)(F_\beta) = 0 .$$

Hence (4.9) holds in this case.

iii) Assume that $\beta(a) = -1$.

$$\delta\Delta(F) = \delta(- [\text{or}(F^a_\beta) : \text{or}(F_\beta)] \cdot F^a_\beta) = - [F^a_\beta : F_\beta] \cdot \sum_{F_{\beta'} \subset F^a_\beta} [F^a_\beta : F_{\beta'}] \cdot F_\beta .$$

$$\Delta\delta(F_\beta) = \Delta\Big(\sum_{F_{\beta''} \subset F_\beta} [F_\beta : F_{\beta''}] \cdot F_{\beta''} \Big) = \sum_{F_{\beta''} \subset F_\beta - [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''}} .$$

$$(\delta\Delta + \Delta\delta)(F_\beta) = - [F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta_{+1}}] \cdot F_{\beta_{+1}} - [F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta_{-1}}] \cdot F_{\beta_{-1}} +$$

$$+ \sum_{\substack{F_{\beta'} \subset F^a_\beta \\ \beta'(a) \neq \pm 1}} - [F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta'}] \cdot F_\beta +$$

$$+ \sum_{F_{\beta''} \subset F_\beta} - [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''} .$$

First note that

$$[F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta_{+1}}] = [\text{or}(F_\beta) : \text{or}(F_{\beta_{+1}})]$$

and that

$$F_{\beta_{-1}} = F_\beta \quad \text{(pointwise)}.$$

Hence

$$(\delta\Delta + \Delta\delta)(F_\beta) = -[\text{or } F_\beta : \text{or } F_{\beta_{+1}}] \cdot F_{\beta_{+1}} - F_\beta +$$
$$+ \sum_{\substack{F_{\beta''} \subset F_\beta \\ \beta'(a) \neq \pm 1}} [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''} +$$
$$+ \sum_{F_{\beta''} \subset F_\beta} [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''} =$$
$$= (i_{+1} \circ C(q) - \mathrm{Id}_{C(I_a(V, V_0))})(F_\beta) +$$
$$+ \sum_{\substack{F_{\beta'} \subset F^a_\beta \\ \beta'(a) \neq \pm 1}} -[F^a_\beta : F_\beta] \cdot [F^a_\beta : F_\beta] \cdot F_{\beta'} +$$
$$+ \sum_{F_{\beta''} \subset F_\beta} [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''} .$$

To prove (4.9) we have to show

$$(4.10) \qquad \sum_{\substack{F_{\beta''} \subset F^a_\beta \\ \beta'(a) \neq \pm 1}} [F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta'}] \cdot F_{\beta'} + \sum_{F_{\beta''} \subset F_\beta} [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''} = 0 .$$

Let $F_{\beta'}$ be a face of F^a_β such that $\beta'(a) \neq 1$. Then $F_{\beta'} = F^a_{\beta'_{-1}}$. Let F_β be a face of F_β. Then $F^a_{\beta''}$ is a face of F^a_β given by $\beta'' \mid B \setminus \{a\} \to \{-1, 1\}$. Note also that the above correspondence is $1 - 1$. Thus $F_{\beta'}$ occurs in our sum with the following coefficients:

$$-[F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta'}] \cdot F_{\beta'} - [F_\beta : F_{\beta'_{-1}}] \cdot [F^a_{\beta'} : F_{\beta'_{-1}}] \cdot F_{\beta'} =$$
$$= -([F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta'}] + [F_\beta : F_{\beta'_{-1}}] \cdot [F_{\beta'} : F_{\beta'_{-1}}]) \cdot F_{\beta'} .$$

Proving Lemma (2.2) we showed:

$$[F^a_\beta : F_\beta] \cdot [F_\beta : F_{\beta'_{-1}}] + [F^a_\beta : F_{\beta'}] \cdot [F_{\beta'} : F_{\beta'_{-1}}] = 0 .$$

Our result is a consequence of the following simple lemma:

(4.11) LEMMA. *Let* $|A| = |B| = |C| = |D| = 1$ *and* $A \cdot B + C \cdot D = 0$. *Then* $A \cdot C + B \cdot D = 0$. □

(4.12) REMARK. All of the chain homomorphisms constructed above commute with augmentations. Hence an augmented complex of a face is also acyclic (look (3.3)).