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Geometria differenziale. — *On Geometries associated with Multiple Integrals*. Nota di EVAN T. DAVIES, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — La presente Nota viene a collegare e completare due lavori anteriori dell'Autore [1, 2] concernenti gli spazi areolari. Dopo aver mostrato (Teorema I, n. 3) come ad ogni funzione su di una varietà differenziabile – che sia integrando di un integrale multiplo invariante –, possa venire associata una connessione soddisfacente a certe due condizioni, con l'uso di questa si esprime (Teorema II, n. 4) la condizione affinché un sottospazio risulti estremale.

1. INTRODUCTION

This paper returns to certain topics under the general heading of Areal Spaces which have been the subject of two recent papers [1, 2] (1). In each of the papers mentioned the two approaches to the problem of getting the essentials of a geometry of a parameter invariant multiple integral have been treated. In one approach the question of finding a two-index metric tensor is avoided [6, 7], while in the other [3, 4] a two index metric tensor is used, and the problem of finding a connection with respect to which the metric is covariant constant is considered. Connection parameters have been obtained corresponding to both approaches, but those obtained for one are not related to those obtained for the other. In this paper it is proved that a symmetric connection can be found satisfying the postulates of Buchin Su on the one hand, and satisfying a Ricci identity for metric tensors on the other. Use is made of four-index metric tensors introduced by Rund.

2. AN IDENTITY FOR THE L FUNCTION

Since our problem is local let us assume that we are in a coordinate neighbourhood of a differentiable manifold and that we have a function L of n variables $x^i (i, j, k, \cdots = 1, \cdots, n)$ and mn variables $p^i_\alpha (\alpha, \beta, \gamma, \cdots = 1, \cdots, m)$ where $1 < m < n$. We assume that the L is sufficiently differentiable and that it can be the integrand function in a parameter-invariant multiple integral. We refer to [5] for details of this section.

If we write ∂_h or $, h$ for $\frac{\partial}{\partial x^h}$ and ∂^α_i or $;^\alpha_i$ for $\partial/\partial p^i_\alpha$ we can introduce

$$(2.1) \qquad g^{\alpha\beta}_{ij} = \frac{2}{m} (L^{2/m}) ;^{\alpha\beta}_{ij}$$

(*) Nella seduta dell'11 novembre 1972.

(1) These refer to the list of papers at the end.

so that the homogeneity properties of the L give

$$\frac{1}{m} g_{ij}^{\alpha\beta}(x, p)\, p_\alpha^i\, p_\beta^j = L^{2/m}. \tag{2.2}$$

Differentiation with respect to the x and the p will give

$$(L^{2/m})_{,k} = \frac{1}{m} g_{ij,k}^{\alpha\beta}\, p_\alpha^i\, p_\beta^j \tag{2.3}$$

$$(L^{2/m})\,;_i^\alpha = \frac{2}{m} g_{ij}^{\alpha\beta}\, p_\beta^j. \tag{2.4}$$

From the definition of the generalized symbols of the first kind

$$2\begin{bmatrix} \alpha\beta \\ ik\,,j \end{bmatrix} = g_{jk,i}^{\alpha\beta} + g_{ij,k}^{\alpha\beta} - g_{ik,j}^{\alpha\beta} \tag{2.5}$$

we deduce from (2.3) that

$$(L^{2/m})_{,k} = \frac{2}{m}\begin{bmatrix} \alpha\beta \\ ik\,,j \end{bmatrix} p_\alpha^i\, p_\beta^j. \tag{2.6}$$

Using the generalized Christoffel symbols of the second kind

$$\begin{Bmatrix} h\alpha \\ ij\gamma \end{Bmatrix} = g_{\gamma\beta}^{hk}\begin{bmatrix} \alpha\beta \\ ij\,,k \end{bmatrix} \tag{2.7}$$

we can modify (2.6) to the form

$$(L^{2/m})_{,k} = \frac{2}{m} g_{jh}^{\beta\gamma}\begin{Bmatrix} h\alpha \\ ik\gamma \end{Bmatrix} p_\alpha^i\, p_\beta^j \tag{2.8}$$

and, using (2.4) this can again be written as

$$(L^{2/m})_{,k} - \begin{Bmatrix} h\alpha \\ ik\gamma \end{Bmatrix} p_\alpha^i\,(L^{2/m})\,;_h^\gamma = 0 \tag{2.9}$$

or, on introducing the notation

$$G_{k\gamma}^h(x, p) = \begin{Bmatrix} h\alpha \\ ik\gamma \end{Bmatrix} p_\alpha^i \tag{2.10}$$

we can finally write

$$L_{,k} - G_{k\alpha}^m\, L\,;_m^\alpha = 0 \tag{2.11}$$

which was the identity in the L required for the next section. The functions G so introduced are homogeneous of degree one in the variables p.

3. The symmetric connection

In the search for suitable geometrical entities based on the function L, use has been made of a two-index metric tensor deducible, by algebraic or transcendental methods, from the L. In the literature three such methods of obtaining such two-index metric tensors have appeared [1, 3, 4]. They all satisfy the condition that if $g_{ij}(x, p)$ denotes the metric tensor, which is homogeneous of degree zero in the p, and if $x^i = x^i(t^\alpha)$ are the parametric equations of a regular portion of a subspace with $p^i_\alpha = \partial x^i/\partial t^\alpha$ in this case so that

$$b_{\alpha\beta} = g_{ij}\, p^i_\alpha\, p^j_\beta \tag{3.1}$$

are the metric coefficients for the m-dimensional subspace, then

$$\det(b_{\alpha\beta}) = \mathrm{L}^2. \tag{3.2}$$

This will give

$$p^\alpha_i \overset{\mathrm{def}}{=} \mathrm{L}^{-1}\, \mathrm{L}\,;\,{}^\alpha_i = b^{\alpha\beta}\, g_{ij}\, p^j_\beta \tag{3.3}$$

giving

$$p^\alpha_i\, p^i_\beta = \delta^\alpha_\beta \tag{3.4}$$

we shall also need

$$\beta^i_j = p^i_\alpha\, p^\alpha_j \quad , \quad \gamma^i_j = \delta^i_j - \beta^i_j. \tag{3.5}$$

If we make a direct calculation we get

$$(\sqrt{b})\,;\,{}^\alpha_i = \sqrt{b}\,\left\{ b^{\alpha\beta}\, g_{ij}\, p^j_\beta + \tfrac{1}{2}\, b^{\gamma\delta}\, g_{mn}\,;\,{}^\alpha_i\, p^m_\gamma\, p^n_\delta \right\} \tag{3.6}$$

so that a comparison with (3.3) gives

$$b^{\gamma\delta}\, g_{mn}\,;\,{}^\alpha_i\, p^m_\gamma\, p^n_\delta = 0. \tag{3.7}$$

From a metric $g_{ij}(x, p)$, a connection which is symmetrical and Euclidean in the sense that the associated covariant derivative of the metric tensor is zero can be obtained by using the notion of an osculating Riemannian space along a curve. This method, which was used successfully in the derivation of a Euclidean connection for Finsler space, and also for areal spaces [1], amounts to using a non-integrable relation of the form

$$dp^i_\alpha + G^i_{k\alpha}(x, p)\, dx^k = 0 \tag{3.8}$$

along a curve, so that a Riemannian space with a metric $\gamma_{ij}(x) = g_{ij}(x, p(x))$ is determined in the neighbourhood of that curve with

$$\gamma_{ij,h} = g_{ij,h} + g_{ij}\,;\,{}^\alpha_m\, p^m_{\alpha,h} \tag{3.9}$$

so that using (3.8) to replace the $p^m_{\alpha,h}$ we have

$$\gamma_{ij,h} = e_h\, g_{ij} \tag{3.10}$$

where

$$e_h = \partial_h - G^m_{h\alpha}\, \partial^\alpha_m. \tag{3.11}$$

If we then introduce

$$(3.12) \qquad 2\,\Gamma_{ijh} = e_i\, g_{jh} + e_j\, g_{ih} - e_h\, g_{ij} \quad , \quad \Gamma^h_{ij} = g^{hk}\,\Gamma_{ijk}$$

the Γ^h_{ij} will be the three-index symbols of the osculating Riemannian space, and hence will have the appropriate transformation laws. From (3.12) we deduce

$$(3.13) \qquad e_i\, g_{kl} = g_{ml}\,\Gamma^m_{ik} + g_{km}\,\Gamma^m_{il}$$

or

$$(3.14) \qquad g_{kl,i} = G^m_{i\alpha}\, g_{kl}\,;\,{}^{\alpha}_{m} + g_{ml}\,\Gamma^m_{ik} + g_{km}\,\Gamma^m_{il}\,.$$

If we now calculate

$$\mathrm{L}_{,k} = (\sqrt{b})_{,k} = \frac{1}{2}\,\mathrm{L} b^{\gamma\delta}\, g_{mn,k}\, p^m_\gamma\, p^n_\delta$$

and use (3.14) and the identity (3.7) it simplifies to

$$(3.15) \qquad \mathrm{L}_{,k} = \mathrm{L} p^n_\gamma\, p^\gamma_m\, \Gamma^m_{nk}$$

which may also be written, on using (3.3) as

$$(3.16) \qquad \mathrm{L}_{,k} - \Gamma^m_{kn}\, p^n_\alpha\, \mathrm{L}\,;\,{}^{\alpha}_{m} = 0$$

which corresponds to (2.11), and which are the so-called „ equations of connection " of Buchin Su [7].

We may therefore state

THEOREM I. *From the fundamental function* L *there can be deduced a set of functions* G *which lead to a connection* Γ *which satisfies*:

(a) *the Ricci identity for the case of a two-index metric tensor*;

(b) *the "equations of connection" corresponding to Postulate* A *of Buchin Su* [7].

4. Minimal Subspaces

We now recall that minimal subspaces are characterized by the vanishing of the Euler–Lagrange expression

$$(4.1) \qquad \mathrm{E}_i = \frac{\mathrm{d}}{\mathrm{d}t^\alpha}\,(\mathrm{L}\,;\,{}^{\alpha}_{i}) - \mathrm{L}\,,\,i$$

an expression which we proceed to express in terms of the connection Γ. Writing (3.14) in the form

$$\mathrm{L}_{,h} = \mathrm{L}\,;\,{}^{\beta}_{j}\,\Gamma^j_{mh}\, p^m_\beta$$

a differentiation with respect to the p^i_α gives

$$(4.2) \qquad \mathrm{L}_{,h}\,;\,{}^{\alpha}_{i} = \mathrm{L}\,;\,{}^{\alpha\beta}_{ij}\,\Gamma^j_{mh}\, p^m_\beta + \mathrm{L}\,;\,{}^{\alpha}_{j}\,\Gamma^j_{ih} + \mathrm{L}\,;\,{}^{\beta}_{j}\,\Gamma^j_{mh}\,;\,{}^{\alpha}_{i}\, p^m_\beta\,.$$

If we write for brevity

$$P^j_{\alpha\beta} = \frac{\partial^2 x^j}{\partial t^\alpha \partial t^\beta} + \Gamma^j_{mn}\, p^m_\alpha\, p^n_\beta$$

we obtain, on using (3.3) and (3.5)

$$E_i = L\,;^{\alpha\beta}_{ij}\, P^j_{\alpha\beta} + L\beta^n_m\, \Gamma^m_{nh}\,;^{\alpha}_{i}\, p^h_\alpha . \tag{4.3}$$

We note that $L\,;^{\alpha\beta}_{ij}$ is related to the so-called Legendre form $L^{\alpha\beta}_{ij}$ by

$$L\,;^{\alpha\beta}_{ij} = LL^{\alpha\beta}_{ij} + p^\alpha_i\, p^\beta_j - p^\alpha_j\, p^\beta_i$$

and if we use

$$L\,\frac{dL}{dt^\alpha} = p^\beta_j\, P^j_{\alpha\beta}$$

we easily verify that

$$E_i = L^{\alpha\beta}_{ij}\, P^j_{\alpha\beta} + L\beta^n_m\, \Gamma^m_{nh}\,;^{\alpha}_{i}\, p^h_\alpha . \tag{4.4}$$

We now wish to prove that the second term on the right-hand side vanishes for the Γ which has been introduced.

We shall need to express the Legendre form in terms of the metric tensors. From the definition (3.3) of p^α_i we obtain, on using (3.5)

$$p^\alpha_i\,;^{\beta}_{j} = b^{\alpha\gamma}\, g_{mn}\,;^{\beta}_{j}\, p^n_\gamma\, \gamma^m_i + b^{\alpha\beta}\, g_{im}\, \gamma^m_i - p^\alpha_j\, p^\beta_i \tag{4.5}$$

from which we can deduce

$$L^{\alpha\beta}_{ij} = b^{\alpha\gamma}\, g_{mn}\,;^{\beta}_{j}\, p^n_\gamma\, \gamma^m_i + b^{\alpha\beta}\, g_{im}\, \gamma^m_j . \tag{4.6}$$

A further use of (4.5) and (4.6) gives us, from (3.5)

$$\beta^k_i\,;^{\beta}_{j} = p^k_\alpha\, L^{\alpha\beta}_{ij} + \gamma^k_j\, p^\beta_i . \tag{4.7}$$

We now proceed to prove that

$$\beta^n_m\, \Gamma^m_{nh}\,;^{\alpha}_{i} = 0 . \tag{4.8}$$

For this we rewrite (3.16) in the form

$$(\log L)_{,h} = \Gamma^j_{mh}\, \beta^m_j$$

so that

$$(\log L)_{,h}\,;^{\alpha}_{i} = \Gamma^j_{mh}\,;^{\alpha}_{i}\, \beta^m_j + \Gamma^j_{mh}\, \beta^m_j\,;^{\alpha}_{i} . \tag{4.9}$$

On the other hand

$$(\log L)\,;^{\alpha}_{i} = p^\alpha_i = b^{\alpha\gamma}\, g_{im}\, p^m_\gamma$$

and $p^\alpha_{i,h}$, on using

$$\frac{\partial b^{\alpha\gamma}}{\partial x^h} = -\, b^{\alpha\zeta}\, b^{\gamma\eta}\, g_{mn,h}\, p^m_\zeta\, p^n_\eta$$

and further using (3.14) and (4.6), gives

$$(\log \mathrm{L})\,;\,{}^{\alpha}_{i,h} = \Gamma^{n}_{hm}\,[p^{m}_{\beta}\,\mathrm{L}^{\alpha\beta}_{in} + p^{\alpha}_{n}\,\gamma^{m}_{i}]$$

which, in turn, on taking account of (4.7) gives

$$(\log \mathrm{L})\,;\,{}^{\alpha}_{i,h} = \Gamma^{j}_{mh}\,\beta^{m}_{j}\,;\,{}^{\alpha}_{i}. \tag{4.10}$$

A comparison of the right-hand sides of (4.9) and (4.10) immediately establishes the validity of (4.8). Hence we may state

THEOREM II. *The connection which satisfies the conditions of Theorem I also satisfies the identity* (4.8), *which implies Postulate* B *of Buchin Su. In terms of this connection therefore the condition for an extremal subspace is expressed by*

$$\mathrm{L}^{\alpha\beta}_{ij}\left(\frac{\partial^2 x^j}{\partial t^\alpha\,\partial t^\beta} + \Gamma^{j}_{mn}\,p^{m}_{\alpha}\,p^{n}_{\beta}\right) = 0.$$

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