Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

H. D. Pande, A. Kumar

Bianchi and Veblen identities for the special pseudo-projective curvature tensor fields

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **53** (1972), n.5, p. 384–388.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_53_5_384_0>

Geometria differenziale. — *Bianchi and Veblen identities for the special pseudo-projective curvature tensor fields.* Nota [(*)] di H. D. PANDE e A. KUMAR presentata dal Socio E. BOMPIANI.

RIASSUNTO. — In uno spazio di Finsler si considerano trasformazioni della funzione integranda dette pseudo-proiettive speciali e si ritrovano gli analoghi degli invarianti di Bianchi e Veblen per i tensori di curvatura trasformati.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n [1] [(1)], in which the Berwald connection coefficient $G^i_{jk}(x, \dot{x})$ is defined by $G^i_{jk} = \dot{\partial}_j \dot{\partial}_k G^i$, where $G^i(x, \dot{x})$ is positively homogeneous of degree two in its directional arguments. The geodesic deviation tensor field H^i_j is given by

$$(1.1) \qquad H^i_k(x, \dot{x}) = 2\,\partial_k G^i - \partial_h \dot{\partial}_k G^i \dot{x}^h + 2\, G^i_{kl} G^l - \dot{\partial}_l G^i \dot{\partial}_k G^l.$$

The curvature tensors and their contractions are

$$(1.2) \qquad H^i_{jk} = 2\,\partial_{[k} \dot{\partial}_{j]} G^i + 2\, G^i_{r[k} \dot{\partial}_{j]} G^r,$$

$$(1.3) \qquad H^i_{hjk} = 2\,\partial_{[k} G^i_{j]h} + 2\, G^r_{h[j} G^i_{k]r} + 2\, G^i_{rh[k} \dot{\partial}_{j]} G^r,$$

$$(1.4) \qquad a)\ H_{ij} = H^l_{ijl}, \qquad b)\ 2\,H_{[jk]} = H^l_{lkj},$$

$$c)\ H_j = H^l_{jl}, \qquad d)\ H = H^i_i/(n-1).$$

These curvature tensors satisfy the following identities:

$$(1.5) \qquad a)\ H^i_{jkh} = -H^i_{jhk}, \qquad b)\ H^j_k \dot{x}^k = 0,$$

$$c)\ H_{jk} \dot{x}^j = H_k, \qquad d)\ \dot{\partial}_r H^r_j \dot{x}^j + (n-1)\,H = 0.$$

The projective deviation tensor field W^i_j is obtained from the deviation tensor field H^i_j and is given by

$$(1.6) \qquad W^j_k(x, \dot{x}) = H^j_k - H\delta^j_k - (\dot{\partial}_i H^i_k - \dot{\partial}_k H)\, \dot{x}^j/(n+1).$$

The projective curvature tensor is defined by $W^j_{ihk} = \frac{2}{3}\, \dot{\partial}_i \dot{\partial}_{[h} W^j_{k]}$ and $W^j_{hk} = \frac{2}{3}\, \dot{\partial}_{[h} W^j_{k]}$.

(*) Pervenuta all'Accademia il 4 ottobre 1972.

(1) Numbers in brackets refer to the references at the end of the paper.
The notations ∂_i and $\dot{\partial}_i$ denotes the operators $\partial/\partial x^i$ and $\partial/\partial \dot{x}^i$ respectively.

Noting that W^i_j is homogeneous of degree two in its directional argument, we have the following simple identities:

$$W^j_{hk}\,\dot{x}^h = W^j_k \tag{1.7}$$

$$W^j_{ihk}\,\dot{x}^i = W^j_{hk} \quad , \quad W^j_{ihk}\,\dot{x}^i\,\dot{x}^h = W^j_k \tag{1.8}$$

$$W^j_k\,\dot{x}^k = 0 \quad , \quad \dot{\partial}_h W^j_k\,\dot{x}^k = -W^j_h \tag{1.9}$$

$$\dot{\partial}_i W^i_k = 0 \quad , \quad W^j_{ihj} = W^j_{hj} = W^j_j = 0 . \tag{1.10}$$

2. Special pseudo–projective tensor field

The projective transformation is defined by

$$\bar{G}^i(x\,,\dot{x}) = G^i(x\,,\dot{x}) - P(x\,,\dot{x})\,\dot{x}^i . \tag{2.1}$$

Where $P(x\,,\dot{x})$ is an arbitrary scalar function, positively homogeneous of the first degree in the $\dot{x}^i$. This change may be regarded as representing a mapping of the two path spaces characterised by the functions G^i and $\bar{G}^i$ respectively on to each other. Let us consider the special projective change [1] characterised by (2.1) for which the function $P(x\,,\dot{x})$ is to satisfy the following relations:

$$P = \frac{1}{(n+1)}\,G^r_{rs}\,\dot{x}^s \qquad \text{or} \qquad \dot{\partial}_k\,\dot{\partial}_h\,P = \frac{1}{(n+1)}\,G^r_{rkh} . \tag{2.2}$$

Sinha [2] has defined the pseudo projective tensor field W^{*i}_j, by taking two scalar functions a and b dependent on $(x\,,\dot{x})$ and homogeneous of degree zero in the $\dot{x}^i$. Here we consider a tensor field as the linear combination of the deviation tensor field H^i_j and projective deviation tensor field W^i_j with the help of function $P(x\,,\dot{x})$ which characterises the special projective change of (2.1). We call this tensor field as the special pseudo–projective tensor field and it is defined by

$$T^i_j(x\,,\dot{x}) = PW^i_j + HH^i_j . \tag{2.3}$$

From this tensor field we obtain the expressions for the special pseudo projective curvature tensor fields T^i_{hj} and $T^i_{lhj}(x\,,\dot{x})$ defined by $T^i_{hj}(x\,,\dot{x}) = \frac{2}{3}\,\dot{\partial}_{[h}\,T^i_{j]}$ and $T^i_{lhj}(x\,,\dot{x}) = \dot{\partial}_l\,T^i_{hj}$: we have

$$T^i_{hj} = PW^i_{hj} + HH^i_{hj} + \frac{2}{3}\,\{\dot{\partial}_{[h}\,PW^i_{j]} + \dot{\partial}_{[h}\,HH^i_{j]}\} \tag{2.4}$$

and

$$T^i_{lhj} = PW^i_{lhj} + HH^i_{lhj} + \dot{\partial}_l\,PW^i_{hj} + \dot{\partial}_l\,HH^i_{hj} + \tag{2.5}$$

$$+ \frac{2}{3}\,[\dot{\partial}^2_{l[h}\,PW^i_{j]} + \dot{\partial}_{[h}\,P\dot{\partial}_{(l)}\,W^i_{j]} + \dot{\partial}^2_{l[h}\,HH^i_{j]} + \dot{\partial}_{[h}\,H\dot{\partial}_{(l)}\,H^i_{j]}]$$

using the homogeneity properties of the functions P, H^i_j and W^i_j we can show that T^i_j are also positively homogeneous of degree two in its directional argument. We prove the following theorems.

THEOREM 2.1. *The Bianchi identities for the special pseudo projective curvature tensor field* T^i_{hj} *and* T^i_{lhj} *are given by* (*)

$$\begin{aligned}(2.6)\qquad & T^i_{hj(k)} + T^i_{jk(h)} + T^i_{kh(j)} = (P_{(k)} W^i_{hj} + P_{(h)} W^i_{jk} + P_{(j)} W^i_{kh}) + \\ & + P\,(W^i_{hj(k)} + W^i_{jk(h)} + W^i_{kh(j)}) + (H_{(k)} H^i_{hj} + H_{(h)} H^i_{jk} + H_{(j)} H^i_{kh}) + \\ & + H\,(H^i_{hj(k)} + H^i_{jk(h)} + H^i_{kh(j)}) + \\ & + \frac{2}{3}\{(\dot\partial_{[h} P_{\langle(k)\rangle} W^i_{j]} + \dot\partial_{[j} P_{\langle(h)\rangle} W^i_{k]} + \dot\partial_{[k} P_{\langle(j)\rangle} W^i_{h]}) + \\ & + (\dot\partial_{[h} PW^i_{j](k)} + \dot\partial_{[j} PW^i_{k](h)} + \dot\partial_{[k} PW^i_{h](j)}) + \\ & + (\dot\partial_{[h} H_{\langle(k)\rangle} H^i_{j]} + \dot\partial_{[j} H_{\langle(h)\rangle} H^i_{k]} + \dot\partial_{[k} H_{\langle(j)\rangle} H^i_{h]}) + \\ & + (\dot\partial_{[h} HH^i_{j](k)} + \dot\partial_{[j} HH^i_{k](h)} + \dot\partial_{[k} HH^i_{h](j)})\}.\end{aligned}$$

and

$$\begin{aligned}(2.7)\qquad & T^i_{lhj(k)} + T^i_{ljk(h)} + T^i_{lkh(j)} = (P_{(k)} W^i_{lhj} + P_{(h)} W^i_{ljk} + P_{(j)} W^i_{lkh}) + \\ & + P\,(W^i_{lhj(k)} + W^i_{ljk(h)} + W^i_{lkh(j)}) + (H_{(k)} H^i_{lhj} + H_{(h)} H^i_{ljk} + H_{(j)} H^i_{lkh}) + \\ & + H\,(H^i_{lhj(k)} + H^i_{ljk(h)} + H^i_{lkh(j)}) + (\dot\partial_l P_{(k)} W^i_{hj} + \dot\partial_l P_{(h)} W^i_{jk} + \dot\partial_l P_{(j)} W^i_{kh}) + \\ & + (\dot\partial_l PW^i_{hj(k)} + \dot\partial_l PW^i_{jk(h)} + \dot\partial_l PW^i_{kh(j)}) + \\ & + (\dot\partial_l H_{(k)} H^i_{hj} + \dot\partial_l H_{(h)} H^i_{jk} + \dot\partial_l H_{(j)} H^i_{kh}) + \\ & + (\dot\partial_l HH^i_{hj(k)} + \dot\partial_l HH^i_{jk(h)} + \dot\partial_l HH^i_{kh(j)}) + Q^i_{lhkj}\end{aligned}$$

where

$$\begin{aligned}(2.8)\qquad & Q^i_{lhkj} \overset{\text{def.}}{=} \frac{2}{3}\{(\dot\partial^2_{l[h} P_{\langle(k)\rangle} W^i_{j]} + \dot\partial^2_{l[j} P_{\langle(h)\rangle} W^i_{k]} + \dot\partial^2_{l[k} P_{\langle(j)\rangle} W^i_{h]}) + \\ & + (\dot\partial^2_{l[h} PW^i_{j](k)} + \dot\partial^2_{l[j} PW^i_{k](h)} + \dot\partial^2_{l[k} PW^i_{h](j)}) + \\ & + (\dot\partial_{[h} P_{\langle(k)\rangle} \dot\partial_{\langle l\rangle} W^i_{j]} + \dot\partial_{[j} P_{\langle(h)\rangle} \dot\partial_{\langle l\rangle} W^i_{k]} + \dot\partial_{[k} P_{\langle(j)\rangle} \dot\partial_{\langle l\rangle} W^i_{h]}) + \\ & + (\dot\partial_{[h} P\dot\partial_{\langle l\rangle} W^i_{j](k)} + \dot\partial_{[j} P\dot\partial_{\langle l\rangle} W^i_{k](h)} + \dot\partial_{[k} P\dot\partial_{\langle l\rangle} W^i_{h](j)}) + \\ & + (\dot\partial^2_{l[h} H_{(k)} H^i_{j]} + \dot\partial^2_{l[j} H_{(h)} H^i_{k]} + \dot\partial^2_{l[k} H_{(j)} H^i_{h]}) + \\ & + (\dot\partial^2_{l[h} HH^i_{j](k)} + \dot\partial^2_{l[j} HH^i_{k](h)} + \dot\partial^2_{l[k} HH^i_{h](j)}) + \\ & + (\dot\partial_{[h} H_{\langle(k)\rangle} \dot\partial_{\langle l\rangle} H^i_{j]} + \dot\partial_{[j} H_{\langle(h)\rangle} \dot\partial_{\langle l\rangle} H^i_{k]} + \dot\partial_{[k} H_{\langle(j)\rangle} \dot\partial_{\langle l\rangle} H^i_{h]}) + \\ & + (\dot\partial_{[h} H\dot\partial_{\langle l\rangle} H^i_{j](k)} + \dot\partial_{[j} H\dot\partial_{\langle l\rangle} H^i_{k](h)} + \dot\partial_{[k} H\dot\partial_{\langle l\rangle} H^i_{h](j)})\}.\end{aligned}$$

(*) Where the index k in the notation $\langle\ \rangle$ is free from the symmetric and skew-symmetric, parts.

Proof. Differentiating covariantly equations (2.4) and (2.5) with respect to $\dot{x}^k$ in the sense of Berwald we obtain,

$$\text{(2.9)}\qquad T^i_{hj(k)} = P_{(k)} W^i_{hj} + PW^i_{hj(k)} + H_{(k)} H^i_{hj} + HH^i_{hj(k)} + $$
$$+ \frac{2}{3} \{(\dot\partial_{[h} P_{\langle(k)\rangle} W^i_{j]} + \dot\partial_{[h} PW^i_{j](k)} + \dot\partial_{[h} H_{\langle(k)\rangle} H^i_{j]} + \dot\partial_{[h} HH^i_{j](k)})\}$$

and

$$\text{(2.10)}\qquad T^i_{lhj(k)} = P_{(k)} W^i_{lhj} + PW^i_{lhj(k)} + H_{(k)} H^i_{lhj} + HH^i_{lhj(k)} +$$
$$+ \dot\partial_l P_{(k)} W^i_{hj} + \dot\partial_l PW^i_{hj(k)} + \dot\partial_l H_{(k)} H^i_{hj} + \dot\partial_l HH^i_{hj(k)} +$$
$$+ \frac{2}{3} \{\dot\partial^2_{l[h} P_{\langle(k)\rangle} W^i_{j]} + \dot\partial^2_{l[h} PW^i_{j](k)} + \dot\partial_{[h} P_{\langle(k)\rangle} \dot\partial_{\langle l\rangle} W^i_{j]} + \dot\partial_{[h} P\dot\partial_{\langle l\rangle} W^i_{j](k)} +$$
$$+ \dot\partial^2_{l[h} H_{\langle(k)\rangle} H^i_{j]} + \dot\partial^2_{l[h} HH^i_{j](k)} + \dot\partial_{[h} H_{\langle(k)\rangle} \dot\partial_{\langle l\rangle} H^i_{j]} + \dot\partial_{[h} H\dot\partial_{\langle l\rangle} H^i_{j](k)}\}.$$

By taking the cyclic permutation of the indices h, j and k in $T^i_{hj(k)}$ and $T^i_{lhj(k)}$ and adding all the terms, we obtain the result.

THEOREM 2.2. *The Veblen identity for the special pseudo-projective curvature tensor field* T^i_{lhj} *is given by,*

$$\text{(2.11)}\qquad T^i_{lhj(k)} + T^i_{jlk(h)} + T^i_{kjh(l)} + T^i_{hkl(j)} =$$
$$= (P_{(k)} W^i_{lhj} + P_{(h)} W^i_{jlk} + P_{(l)} W^i_{kjh} + P_{(j)} W^i_{hkl}) +$$
$$+ P\,(W^i_{lhj(k)} + W^i_{jlk(h)} + W^i_{kjh(l)} + W^i_{hkl(j)}) +$$
$$+ (H_{(k)} H^i_{lhj} + H_{(h)} H^i_{jlk} + H_{(l)} H^i_{kjh} + H_{(j)} H^i_{hkl}) +$$
$$+ H\,(H^i_{lhj(k)} + H^i_{jlk(h)} + H^i_{kjh(l)} + H^i_{hkl(j)}) +$$
$$+ (\dot\partial_l P_{(k)} W^i_{hj} + \dot\partial_j P_{(h)} W^i_{lk} + \dot\partial_k P_{(l)} W^i_{jh} + \dot\partial_h P_{(j)} W^i_{kl}) +$$
$$+ (\dot\partial_l PW^i_{hj(k)} + \dot\partial_j PW^i_{lk(h)} + \dot\partial_k PW^i_{jh(l)} + \dot\partial_h PW^i_{kl(j)}) +$$
$$+ (\dot\partial_l H_{(k)} H^i_{hj} + \dot\partial_j H_{(h)} H^i_{lk} + \dot\partial_k H_{(l)} H^i_{jh} + \dot\partial_h H_{(j)} H^i_{kl}) +$$
$$+ (\dot\partial_l HH^i_{hj(k)} + \dot\partial_j HH^i_{lk(h)} + \dot\partial_k HH^i_{jh(l)} + \dot\partial_h HH^i_{kl(j)}) + R^i_{lhkj}$$

where

$$\text{(2.12)}\qquad R^i_{lhkj} \overset{\text{def.}}{=} \frac{2}{3} \{(\dot\partial^2_{l[h} P_{\langle(k)\rangle} W^i_{j]} + \dot\partial^2_{j[l} P_{\langle(h)\rangle} W^i_{k]} + \dot\partial^2_{k[j} P_{\langle(l)\rangle} W^i_{h]} + \dot\partial^2_{h[k} P_{\langle(j)\rangle} W^i_{l]}) +$$
$$+ (\dot\partial^2_{l[h} PW^i_{j](k)} + \dot\partial^2_{j[l} PW^i_{k](h)} + \dot\partial^2_{k[j} PW^i_{h](l)} + \dot\partial^2_{h[k} PW^i_{l](j)}) +$$
$$+ (\dot\partial_{[h} P_{\langle(k)\rangle} \dot\partial_{\langle l\rangle} W^i_{j]} + \dot\partial_{[l} P_{\langle(h)\rangle} \dot\partial_{\langle j\rangle} W^i_{k]} + \dot\partial_{[j} P_{\langle(l)\rangle} \dot\partial_{\langle k\rangle} W^i_{h]} + \dot\partial_{[k} P_{\langle(j)\rangle} \dot\partial_{\langle h\rangle} W^i_{l]}) +$$
$$+ (\dot\partial_{[h} P\dot\partial_{\langle l\rangle} W^i_{j](k)} + \dot\partial_{[l} P\dot\partial_{\langle j\rangle} W^i_{k](h)} + \dot\partial_{[j} P\dot\partial_{\langle k\rangle} W^i_{h](l)} + \dot\partial_{[k} P\dot\partial_{\langle h\rangle} W^i_{l](j)}) +$$
$$+ (\dot\partial^2_{l[h} H_{\langle(k)\rangle} H^i_{j]} + \dot\partial^2_{j[l} H_{\langle(h)\rangle} H^i_{k]} + \dot\partial^2_{k[j} H_{\langle(l)\rangle} H^i_{h]} + \dot\partial^2_{h[k} H_{\langle(j)\rangle} H^i_{l]}) +$$
$$+ (\dot\partial^2_{l[h} HH^i_{j](k)} + \dot\partial^2_{j[l} HH^i_{k](h)} + \dot\partial^2_{k[j} HH^i_{h](l)} + \dot\partial^2_{h[k} HH^i_{l](j)}) +$$
$$+ (\dot\partial_{[h} H_{\langle(k)\rangle} \dot\partial_{\langle l\rangle} H^i_{j]} + \dot\partial_{[l} H_{\langle(h)\rangle} \dot\partial_{\langle j\rangle} H^i_{k]} + \dot\partial_{[j} H_{\langle(l)\rangle} \dot\partial_{\langle k\rangle} W^i_{h]} + \dot\partial_{[k} H_{\langle(j)\rangle} \dot\partial_{\langle h\rangle} H^i_{l]}) +$$
$$+ (\dot\partial_{[h} H\dot\partial_{\langle l\rangle} H^i_{j](k)} + \dot\partial_{[l} H\dot\partial_{\langle j\rangle} H^i_{k](h)} + \dot\partial_{[j} H\dot\partial_{\langle k\rangle} H^i_{h](l)} + \dot\partial_{[k} H\dot\partial_{\langle h\rangle} H^i_{l](j)})\}.$$

Proof. It follows the pattern of the proof of Theorem 2.1.

THEOREM 2.3. *We prove the following identities*:

$$(2.13)\quad T_{lkhj} + T_{kljh} = P\,(W_{lkhj} + W_{kljh}) + (H_{lkhj} + H_{kljh}) + (\dot\partial_l\, PW_{hkj} + \dot\partial_k\, PW_{jlh}) + (\dot\partial_l\, HH_{hkj} + \dot\partial_k\, HH_{jlh}) + \frac{2}{3}\{(\dot\partial^2_{l[h}\, PW_{j]k} + \dot\partial^2_{k[j}\, PW_{h]l}) + (\dot\partial_{[h}\, P\dot\partial_{\langle l\rangle}\, W_{j]k} + \dot\partial_{[j}\, P\dot\partial_{\langle k\rangle}\, W_{h]l}) + (\dot\partial^2_{l[h}\, HH_{j]k} + \dot\partial^2_{k[j}\, HH_{h]l}) + (\dot\partial_{[h}\, H\dot\partial_{\langle l\rangle}\, H_{j]k} + \dot\partial_{[j}\, H\dot\partial_{\langle k\rangle}\, H_{h]l})\},$$

and,

$$(2.14)\quad T_{lkhj} + T_{jhkl} = P\,(W_{lkhj} + W_{jhkl}) + H\,(H_{lkhj} + H_{jhkl}) + (\dot\partial_l\, PW_{hkj} + \dot\partial_j\, PW_{khl}) + (\dot\partial_l\, HH_{hkj} + \dot\partial_j\, HH_{khl}) + \frac{2}{3}\{(\dot\partial^2_{l[h}\, PW_{j]k} + \dot\partial^2_{j[k}\, PW_{l]h}) + (\dot\partial_{[h}\, P\dot\partial_{\langle l\rangle}\, W_{j]k} + \dot\partial_{[k}\, P\dot\partial_{\langle j\rangle}\, W_{l]h}) + (\dot\partial^2_{l[h}\, HH_{j]k} + \dot\partial^2_{j[k}\, HH_{l]h}) + \dot\partial_{[h}\, H\dot\partial_{\langle l\rangle}\, H_{j]k} + \dot\partial_{[k}\, HH_{\langle j\rangle}\, H_{l]h})\}.$$

Proof. With the help of equation (2.5) we get

$$(2.15)\quad T_{lkhj} \overset{\text{def.}}{=} g_{ik}\, T^i_{lhj} = PW_{lkhj} + HH_{lkhj} + \dot\partial_l\, PW_{hkj} + \dot\partial_l\, HH_{hkj} + \frac{2}{3}\{\dot\partial^2_{l[h}\, PW_{j]k} + \dot\partial_{[h}\, P\dot\partial_{\langle l\rangle}\, W_{j]k} + \dot\partial^2_{l[h}\, HH_{j]k} + \dot\partial_{[h}\, H\dot\partial_{\langle l\rangle}\, H_{j]k}\}.$$

Using equation (2.15) we easily obtain the identities (2.13) and (2.14).

REFERENCES

[1] H. RUND, *Differential Geometry of Finsler Spaces*, Springer Verlag, Berlin 1959.

[2] B. B. SINHA and S. P. SINGH, *On Pseudo-Projective Tensor Field Tensor*, N.S., *22*, 317–320 (1971).

[3] H. D. PANDE, *The projective transformation in a Finsler Space*, « Atti Accad. Naz. Lincei, Rend. » (6), *48*, 480–484 (1967).

[4] R. S. MISHRA and H. D. PANDE, *The Ricci Identity*, « Annali di Matematica Pura ed Appl. », (IV), *75*, 355–362 (1967).