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Solution to a Nonlinear Differential Equation With Application to Thomas-Fermi Equations

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Equazioni differenziali. — *Solution to a Nonlinear Differential Equation With Application to Thomas-Fermi Equations.* Nota di JAMES L. REID, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Otteniamo l'equazione differenziale non–lineare del 2° ordine che è soddisfatta dalla funzione $y = [\pm bmu^j v^n]^{k/m}$, $m = j + n$. Le funzioni u e v sono soluzioni della equazione lineare $y'' + r(x) y' + q(x) y = 0$; la b è una costante qualunque; gli esponenti sono reali e non zero. Mostriamo come si possa ottenere, dalla nostra, la equazione di Thomas–Fermi e altre equazioni simili.

J. M. Thomas [1] has shown that the nonlinear differential equation

$$(1) \qquad y'' + r(x) y' + kq(x) y = (1 - l) y'^2 y^{-1} - 1/4\, k W^2 y^{1-4l}, \; (') = d/dx,$$

possesses the solution

$$(2) \qquad y = [u(x) v(x)]^{k/2}, \; kl = 1,$$

where $u(x)$ and $v(x)$ satisfy the linear homogeneous differential equation

$$(3) \qquad y'' + r(x) y' + q(x) y = 0 .$$

The function $W \leq uv' - vu'$ is the Wronskian of u and v, while the number k is assumed real and non-zero. Equation (1) is in the form recorded by Herbst [2].

The object of this Note is twofold. First, we generalize the equation of J. M. Thomas. Next, from this generalization we develop a class of Thomas–Fermi equations and an elementary solution. Also we note that our result may be applied to certain Emden–Fowler equations.

To generalize (1) we assume a function of the form

$$(4) \qquad y = (\pm bmu^j v^n)^{k/m}, \; m = j + n, \quad b = \text{const.},$$

where j, m, and n are assumed real and non-zero, and substitute into the nonlinear equation

$$(5) \qquad y'' + r(x) y' + kq(x) y = R(x, y, y'),$$

where $R(x, y, y')$ represents the nonlinear term to be determined. For convenience we define

$$(6) \qquad A \equiv \pm b (ju^{j-1} u' v^n + nu^j v^{n-1} v')$$

(*) Nella seduta dell'11 novembre 1972.

and express y' and y'' in terms of A and A', i.e.,

(7 a) $$y' = ky^{1-ml}\,\mathrm{A}$$

(7 b) $$y'' = k\,(l - m)\,y^{1-2\,ml}\,\mathrm{A}^2 + ky^{1-lm}\,\mathrm{A}' \,.$$

Thus (5) may be written as

(8) $$k\,(l - m)\,\mathrm{A}^2 + ky^{ml}\,[\mathrm{A}' + r\,(x)\,\mathrm{A} + q\,(x)\,y^{ml}] = y^{2\,ml-1}\,\mathrm{R}\,.$$

In reducing (8) we make use of the assumption that u and v satisfy (3) and of the expression

(9) $$\mathrm{A} = ly^{ml-1}\,y'\,.$$

The remaining details of this calculation are straightforward and (8) reduces to our generalization:

(10) $$y'' + r(x)\,y' + kq(x)\,y = (1 - l)\,y'^2\,y^{-1} - b^2\,knj\,u^{2j-2}\,v^{2n-2}\,\mathrm{W}^2\,y^{1-2ml}\,,$$

which is satisfied by (4).

The Wronskian $\mathrm{W}\,(x)$ can also be expressed by

(11) $$\mathrm{W}\,(x) = \mathrm{W}_0\,w\,(x)\,,\; w\,(x) = \exp\left[-\int_{x_0}^{x} r\,(t)\,\mathrm{d}t\right],$$

where $\mathrm{W}_0 = \mathrm{W}\,(x_0) = \text{constant} \neq 0$. With this notation, (10) may assume the form

(12) $$y'' + r\,(x)\,y' + kq\,(x)\,y = (1 - l)\,y'^2\,y^{-1} - b^2\,u^{2j-2}\,v^{2n-2}\,w^2\,y^{1-2ml}\,,$$

with the solution

(13) $$y = \left[\frac{mbu^j\,v^n}{\pm\,(kjn)^{1/2}\,\mathrm{W}_0}\right]^{k/m}.$$

For real b, $u\,(x)$, $v\,(x)$, we see that only complex solutions are possible when the product kjn is negative. When that product is positive, solutions occur in real or complex form, depending on the choice of sign on the radical (assuming that b, u, and v are real). To emphasize this situation, we retain the $\pm$ sign in solutions (4) and (13). Analogous statements can be made when the sign of b^2 in (12) is reversed by writing the radical as $(-\,kjn)^{1/2}$.

An interesting particular case of (12) follows by setting $r\,(x) \equiv 0$, $l = 1$, $m = 2$, i.e.,

(14) $$y'' + q\,(x)\,y = -\,b^2\,(u/v)^{2j-2}\,y^{-3}\,,$$

the solution being

(15) $$y = \left[\frac{2\,b}{\pm\,[j\,(2 - j)]^{1/2}\,\mathrm{W}_0}\right]^{1/2}\left(\frac{u}{v}\right)^{j/2} v\,,$$

where n has been eliminated in favor of j. The nonlinear differential equation noted by Pinney [3] is a special case of (14) with $j = 1$.

We turn now to the second objective of this Note and consider the case when both $r(x)$ and $q(x)$ are identically zero. Here u and v must be solutions of the simple linear equation $y''=0$, and we choose

$$u = x + c \quad , \quad v = 1 \quad , \quad W = 1, \tag{16}$$

where c is an arbitrary constant. The nonlinear equation that results from (12) may be expressed as

$$y'' = (1 - l)\, y'^2\, y^{-1} + b^2 (x + c)^{2j-2}\, y^{1-2ml} \tag{17}$$

having the elementary particular solution

$$y = \left[\frac{mb}{\pm [kj (j - m)]^{1/2}}\right]^{k/m} (x + c)^{jk/m}, \tag{18}$$

where $j \neq m$. For the values $l = 1$, $m = -1/4$, $j = 3/4$, $b = 1$, $c = 0$, in (17) and (18), we obtain, respectively, the Thomas–Fermi equation

$$y'' = x^{-1/2}\, y^{3/2} \tag{19}$$

and the well known particular solution

$$y = 144\, x^{-3}. \tag{20}$$

We emphasize that the generalization represented by (10) leads directly and simply to the Thomas–Fermi equation and the particular solution (20). This solution was first obtained by L. H. Thomas [4] in a rather laborious analysis.

The question of the existence and uniqueness of a solution of the second order differential equation $y = \varphi(x, y, y')$ subject to the boundary conditions

$$y(0) = y_0 \geq 0 \quad , \quad y(+\infty) = 0 \tag{21}$$

has been discussed by Scorza–Dragoni [5] and Mambriani [6]. Applying Scorza-Dragoni's theorem [5] to (17), we find that a unique solution subject to the boundary conditions (21) will exist for $l = 1$, $c \geq 0$, $m < 0$, and $j > 1/2$. For these parameter values, we may consider (17) as a generalization of the Thomas–Fermi equation.

The parameters of (17) can be adjusted to match those of the Emden–Fowler equation [7]

$$y'' = \pm\, x^{1-M}\, y^{M} \tag{22}$$

by taking $j = (3 - M)/2$, $m = (1 - M)/2$, $b = 1$, $c = 0$, $l = 1$. A particular solution of Eq. (22) is thus

$$y = \left[\frac{1 - M}{\pm [2(3 - M)]^{1/2}}\right]^{\frac{2}{1-M}} x^{(3-M)/(1-M)}. \tag{23}$$

This solution is complex for $M > 3$, is real for $1 < M < 3$ by choosing the negative sign, is real for $M < 1$ by choosing the positive sign, and is not defined at $M = 1,3$.

The equations and particular solutions displayed in this note should be of value in the study of Thomas–Fermi and Emden–Fowler type of differential equations. For detailed analyses and additional references, see the recent papers of Hille [7, 8, 9].

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