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On an extension of an integral equation considered by Volterra and Picone

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Analisi matematica. — *On an extension of an integral equation considered by Volterra and Picone* (*). Nota di MEHMET NAMIK OĞUZTÖRELI, presentata (**) dal Socio M. PICONE.

RIASSUNTO. — L'Autore considera in ciò che segue un'estensione di un'equazione integrale di prima specie del tipo di Volterra–Picone e generalizza certi risultati di Volterra e Picone stabiliti nei Loro oramai classici lavori [1] e [2].

1. In one of his 1897 Memoires [1], V. Volterra considered the integral equation

$$\int_{\alpha x}^{x} K(x, \xi)\, u(\xi)\, d\xi = f(x) - f(0), \tag{1}$$

where α is a constant such that $|\alpha| < 1$, $f(x)$ and $K(x, \xi)$ are certain given functions which are continuously differentiable in x and continuous in ξ for $|x| \leq \gamma$ and $|\xi| \leq x$, and $u(x)$ is an unknown function defined in $|x| < \gamma$. In 1910, M. Picone presented in his elegant paper [2] a direct, short and unified proof for the main results of V. Volterra. It is shown in [1] and [2] that, if $K(x, x) \neq 0$ for $|x| < \gamma$, then the solution of Eq. (1) is equivalent to the solution of the integro–functional equation

$$u(x) = \varphi(x) + \alpha\lambda(x)\, u(\alpha x) + \int_{\alpha x}^{x} H(x, \xi)\, u(\xi)\, d\xi, \tag{2}$$

where

$$\varphi(x) = \frac{f'(x)}{K(x, x)}, \quad \lambda(x) = \frac{K(x, \alpha x)}{K(x, x)}, \quad H(x, \xi) = -\frac{\dfrac{\partial K(x, \xi)}{\partial x}}{K(x, x)}. \tag{3}$$

Eqs. (1) and (2) are closely connected with the following functional equations:

$$\int_{\theta(x)}^{x} K(x, \xi)\, u(\xi)\, d\xi = f(x) - f(0) \tag{4}$$

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and

$$u(x) = \varphi(x) + \lambda(x)\, u[\theta(x)] + \int_0^x \mathrm{H}(x, \xi)\, u(\xi)\, d\xi,$$

(5) $$u(x) = \varphi(x) + \lambda(x)\, u[\theta(x)] + \int_{-x}^x \mathrm{H}(x, \xi)\, u[\psi(\xi)]\, d\xi,$$

$$u(x) = \varphi(x) + \lambda(x)\, u[\theta(x)] + \int_0^a \mathrm{H}(x, \xi)\, u[\psi(\xi)]\, d\xi,$$

where $\theta(x)$ and $\psi(x)$ are given continuous functions such that $|\theta(x)| \leq |x|$ and $|\psi(x)| \leq |x|$ for $|x| \leq \gamma$. Some of Eqs. (4), (5) with certain θ and ψ considered by V. Volterra and M. Picone in [1] and [2], and by E. Picard [3], T. Lalescu [4], A. Myller [5], C. Popovici [6] and J. D. Tamarkin [7]. For certain extensions of the above equations we mention the work of R. Badescu [8], N. Cioranescu [9], B. Colombo [10] and M. Ghermanescu [11]. For ordinary q–difference equations, we refer to the papers of R. C. Adams [12 f, g], G. D. Birkhoff [13], M. G. Carman [14], R. D. Carmichael [15], P. Flamant [16], A. Grevy [17], T. E. Mason [18] and P. Nalli [19].

In the present paper, we consider the extension of the integral equation (1) into several variables

(6) $$\int_{\alpha_1 x_1}^{x_1} d\xi_1 \int_{\alpha_2 x_2}^{x_2} d\xi_2 \cdots \int_{\alpha_n x_n}^{x_n} \mathrm{K}(x_1, x_2, \cdots, x_n; \xi_1, \xi_2, \cdots, \xi_n)\, u(\xi_1, \xi_2, \cdots, \xi_n)\, d\xi_n = f(x_1, x_2, \cdots, x_n)$$

where α_i's are constants such that $|\alpha_i| < 1$, f and K are certain given functions having continuous derivatives $\partial_n f$ and $\partial_n \mathrm{K}$ in $|x_i| \leq \gamma_i$, $|\xi_i| \leq |x_i|$, u is the unknown function defined in $|x_i| < \gamma_i$ $(i = 1, \cdots, n)$, where

(7) $$\partial_n = \frac{\partial^n}{\partial x_1\, \partial x_2 \cdots \partial x_n}.$$

We assume that K is continuous with respect to $\xi_1, \cdots, \xi_n$ for $|x_i| < \gamma_i$ and $|\xi_i| \leq |x_i|$ and

(8) $$f(x_1, \cdots, x_n)\Big|_{x_i = 0} = 0 \qquad (i = 1, \cdots, n).$$

To simplify our presentation we shall restrict ourselves to the case $n = 2$. The general case can be treated similarly.

2. Consider the integral equation

$$(9)\qquad \int_{\alpha x}^{x}\int_{\beta y}^{y} K(x, y; \xi, \eta)\, u(\xi, \eta)\, d\xi\, d\eta$$
$$= f(x, y) - f(x, 0) - f(0, y) - f(0, 0),$$

where α and β are constants such that $|\alpha| < 1$, $|\beta| < 1$, $f(x, y)$ and $K(x, y; \xi, \eta)$ are given functions continuous in $\bar{S}_2$ and $\bar{S}_4$, and $u(x, y)$ is the unknown function defined in S_2, where

$$(10)\qquad \begin{aligned} S_2 &\equiv \{(x, y) \,|\, |x| < \gamma, |y| \le \gamma\} \\ S_4 &\equiv \{(x, y; \xi, \eta) \,|\, |\xi| \le |x|, |\eta| \le |y|, (x, y) \in S_2\} \end{aligned}$$

and $\bar{S}_i$ denotes the closure of S_i. We assume that $\dfrac{\partial^2 f}{\partial x\, \partial y}$ and $\dfrac{\partial^2 K}{\partial x\, \partial y}$ exist and are continuous on $\bar{S}_2$ and $\bar{S}_4$, respectively. We further assume that

$$(11)\qquad K(x, y; x, y) \neq 0 \quad , \quad (x, y) \in \bar{S}_2 .$$

We now differentiate both sides of Eq. (9) successively with respect to x and y. We then obtain the following extension of the integro-functional equation (2):

$$(12)\qquad \begin{aligned} u(x, y) &= \varphi(x, y) + \alpha\lambda(x, y)\, u(\alpha x, y) \\ &+ \beta\mu(x, y)\, u(x, \beta y) + \alpha\beta\nu(x, y)\, u(\alpha x, \beta y) \\ &+ \alpha\int_{\beta y}^{y} H_1(x, y; \eta)\, u(\alpha x, y)\, d\eta + \beta\int_{\alpha x}^{x} H_2(x, y; \xi)\, u(\xi, \beta y)\, d\xi \\ &+ \int_{\beta y}^{y} H_3(x, y; \eta)\, u(x, \eta)\, d\eta + \int_{\alpha x}^{x} H_4(x, y; \xi)\, u(\xi, y)\, d\xi \\ &+ \int_{\alpha x}^{x}\int_{\beta y}^{y} H_0(x, y; \xi, \eta)\, u(\xi, \eta)\, d\xi\, d\eta , \end{aligned}$$

where

$$(13)\qquad \lambda(x, y) = \frac{K(x, y; \alpha x, y)}{K(x, y; x, y)} \quad , \quad \mu(x, y) = \frac{K(x, y; x, \beta y)}{K(x, y; x, y)} ,$$
$$\nu(x, y) = -\frac{K(x, y; \alpha x, \beta y)}{K(x, y; x, y)}$$

and

$$H_0(x, y; \xi, \eta) = -\frac{\partial^2 K(x, y; \xi, \eta)}{\partial x \partial y} / K(x, y; x, y),$$

$$H_1(x, y; \eta) = \frac{\partial K(x, y; \alpha x, \eta)}{\partial y} / K(x, y; x, y),$$

$$(14) \qquad H_2(x, y; \xi) = \frac{\partial K(x, y; \xi, \beta y)}{\partial x} / K(x, y; x, y),$$

$$H_3(x, y; \eta) = -\frac{\partial K(x, y; x, \eta)}{\partial y} / K(x, y; x, y),$$

$$H_4(x, y; \xi) = -\frac{\partial K(x, y; \xi, y)}{\partial x} / K(x, y; x, y).$$

The functions defined by Eqs. (13) and (14) are all continuous in $\bar{S}_4$, and we have

$$(15) \qquad \lambda(0, 0) = \mu(0, 0) = -\nu(0, 0) = 1.$$

Thus, any solution of the integral equation (9) satisfies necessarily the integro-functional equation (12) if f and K satisfy the above mentioned regularity conditions. In the next sections we investigate the solution of Eq. (12).

3. We begin our discussion with the solution of the partial q–difference equation

$$(16) \qquad v(x, y) = \varphi(x, y) + \alpha\lambda(x, y)\, v(\alpha x, y) + \beta\mu(x, y)\, v(x, \beta y) + \\ + \alpha\beta\nu(x, y)\, v(\alpha x, \beta y).$$

The analytic theory of linear partial q–difference equations have been developed by R. C. Adams [12 a, b, c, d, e].

Eq. (16) can be solved by the method of successive approximations. To do so we put

$$v_0(x, y) = \varphi(x, y)$$

$$(17) \qquad v_{n+1}(x, y) = \varphi(x, y) + \alpha\lambda(x, y)\, v_n(\alpha x, y) + \beta\mu(x, y)\, v_n(x, \beta y) \\ + \alpha\beta\nu(x, y)\, v_n(\alpha x, \beta y).$$

To show that the sequence $\{v_n(x, y)\}$ is uniformly convergent in $\bar{S}_2$, it suffices to prove the uniform convergence of the series

$$(18) \qquad v_0(\alpha^i x, \beta^j y) + \sum_{n=0}^{\infty} \{v_{n+1}(\alpha^i x, \beta^j y) - v_n(\alpha^i x, \beta^j y)\}$$

for any $i, j = 0, 1, 2, \cdots$. For this purpose, we put

$$p = |\alpha| \quad , \quad q = |\beta| ,$$

$$M_0 = \max_{\bar{S}_2} |\varphi(x, y)| \quad , \quad M_n = \max_{\bar{S}_2} |v_{n+1}(x, y) - v_n(x, y)| ,$$

(19)
$$\lambda_{i,j} = \max_{\bar{S}_2} |\lambda(\alpha^i x, \beta^j y)| \quad , \quad \mu_{i,j} = \max_{\bar{S}_2} |\mu(\alpha^i x, \beta^j y)| ,$$

$$\nu_{i,j} = \max_{\bar{S}_2} |\nu(\alpha^i x, \beta^j y)| ,$$

$$H_k = \max_{\bar{S}_4} |H_k(x, y; \xi, \eta)| \qquad (k = 0, 1, \cdots, 4).$$

By virtue of Eqs. (15), we have

$$\lim_{i,j\to\infty} \lambda_{i,j} = \lim_{i,j\to\infty} \mu_{i,j} = \lim_{i,j\to\infty} \nu_{i,j} = 1 , \tag{20}$$

and by Eqs. (17)–(19),

$$M_{n+1} \leq \rho_{i,j} M_n \leq \rho_{i,j}^n M_1 \qquad (i, j, n = 0, 1, 2, \cdots) \tag{21}$$

where

$$\rho_{i,j} = p\lambda_{i,j} + q\mu_{i,j} + pq\nu_{i,j} . \tag{22}$$

Note that, on account of Eqs. (20), we have

$$\lim_{i,j=\infty} \rho_{i,j} = \rho \equiv p + q + pq . \tag{23}$$

It follows from the inequalities (21) that the series (18) is dominated by the series $M_0 + M_1 \sum_{n=1}^{\infty} \rho_{i,j}^n$ $(i, j = 0, 1, 2, \cdots)$. Hence, if

$$0 \leq \rho_{i,j} \leq \theta \qquad (0 < \theta < 1), \tag{24}$$

where θ is a constant, then by the M–test of Weierstrass, the series (18) converges absolutely and uniformly to a (continuous) function $v(\alpha^i x, \beta^j y)$ in $\bar{S}_2$. Clearly the limit function $v(x, y)$ is the unique solution of Eq. (16). Note that

$$\max_{\bar{S}_2} |v(\alpha^i x, \beta^j y)| \leq M_0 + \frac{M_1}{1-\theta} \qquad (i, j = 0, 1, 2, \cdots). \tag{25}$$

4. We now consider the integro-functional equation (12). This equation can be solved by the method of successive approximations. To this end, we put

$$u_0(x, y) = v(x, y)$$

$$u_n(x, y) = \varphi_n(x, y) + \alpha\lambda(x, y)\, u_n(\alpha x, y) + \beta\mu(x, y)\, u_n(x, \beta y) + \alpha\beta\nu(x, y)\, u_n(\alpha x, \beta y)$$

$$\varphi_{n+1}(x, y) = \int_{\alpha x}^{x}\int_{\beta y}^{y} H_0(x, y; \xi, \eta)\, u_n(\xi, \eta)\, d\xi\, d\eta + \alpha\int_{\beta y}^{y} H_1(x, y; \eta)\, u_n(\alpha x, \eta)\, d\eta + \beta\int_{\alpha x}^{x} H_2(x, y; \xi)\, u_n(\xi, \beta y)\, d\xi + \int_{\beta y}^{y} H_3(x, y; \eta)\, u_n(x, \eta)\, d\eta + \int_{\alpha x}^{x} H_4(x, y; \xi)\, u_n(\xi, y)\, d\xi. \tag{26}$$

We shall show that the sequence $\{u_n(x, y)\}$ converges uniformly on $\bar{S}_2$. Indeed, consider the series

$$u_0(x, y) + \sum_{n=0}^{\infty}\{u_{n+1}(x, y) - u_n(x, y)\} \tag{27}$$

and put

$$K_n = \max_{\bar{S}_2} |\varphi_{n+1}(x, y) - \varphi_n(x, y)|, \qquad L_n = \max_{\bar{S}_2} |u_{n+1}(x, y) - u_n(x, y)|. \tag{28}$$

Clearly, the second equation in (26) admits a unique continuous solution in $\bar{S}_2$ by virtue of the results of § 3, if $0 < \theta < 1$. Further, from Eqs. (26), we have

$$L_n \leq \frac{K_n}{1-\theta}, \qquad K_n \leq \sigma L_{n-1}, \tag{29}$$

where

$$\sigma = (1-\alpha)(1-\beta)\gamma^2 H_0 + p(1-\beta)\gamma H_1 + q(1-\alpha)\gamma H_2 + (1-\beta)\gamma H_3 + (1-\alpha)\gamma H_4. \tag{30}$$

Hence, we have

$$L_n \leq \left(\frac{\sigma}{1-\theta}\right)^n L_0 \qquad (n = 1, 2, \cdots). \tag{30}$$

Thus, if $\frac{\sigma}{1-\theta} < 1$, the series (27) converges absolutely and uniformly to a continuous function $u(x, y)$ which is the unique solution of Eq. (12).

5. We now illustrate the above analysis with the following kernel

(31) $$K(x, y; \xi, \eta) = 1 - \lambda(x - \xi)(y - \eta)(\alpha x - \xi)(\beta y - \eta) K_1(x, y; \xi, \eta)$$

where K_1, is of class C^2 in $\bar{S}_4$. Then we have $\lambda(x, y) = \mu(x, y) = -\nu(x, y) = 1$, $H_1 = H_2 = H_3 = H_4 \equiv 0$. Hence the integral equation (9) with the above kernel leads us to the integro-functional equation

(32) $$u(x, y) = \varphi(x, y) + \alpha u(\alpha x, y) + \beta u(x, \beta y) - \alpha\beta u(\alpha x, \beta y) + \lambda \int_{\alpha x}^{x} \int_{\beta y}^{y} H_0(x, y; \xi, \eta) u(\xi, \eta) \, d\xi \, d\eta.$$

The partial q-difference equation

(33) $$u(x, y) = \varphi(x, y) + \alpha u(\alpha x, y) + \beta u(x, \beta y) - \alpha\beta u(\alpha x, \beta y)$$

admits the solution

(34) $$u(x, y) = \sum_{m,n=0}^{\infty} \alpha^m \beta^n \varphi(\alpha^m x, \beta^n y).$$

It can be easily verified, by using the successive approximations scheme in § 4, that the functional equation (32) admits a unique solution if

(35) $$|\lambda| < \frac{(1+p)(1+q)}{H\gamma^2},$$

where $H = \max_{\bar{S}_4} H(x, y; \xi, \eta)$, and this solution is analytic in λ in the interval (35).

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