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Algebra topologica. — *Function Algebras over Valued Fields and Measures.* Nota IV (*) (**) di EDWARD BECKENSTEIN, GEORGE BACHMAN e LAWRENCE NARICI, presentata dal Corrisp. G. ZAPPA.

RIASSUNTO. — Vedi la Nota III, dallo stesso titolo, di cui la presente è il completamento.

3. THE HOMOMORPHISMS OF C(T, F) INTO F

In this section we show in our principal result that if F is a complete discretely valued field such that the residue class field V/P of F has nonmeasurable cardinal, then the kernels of the nontrivial homomorphisms of C(T, F) into F are in 1 — 1 correspondence with the regular σ–smooth monotone 0 — 1 measures on the Baire sets of T or equivalently, the maximal ideals M^{μ} such that $Z(M^{\mu})$ is closed under the formation of denumerable intersections. We note that all known cardinals are nonmeasurable and indeed the nonmeasurable cardinals are a large collection [2, p. 161].

Example 1 below demonstrates that a complete discretely valued field F can be constructed having a residue class field of arbitrarily large cardinality. Using this we show that the principal result of this section is true for all complete discretely valued fields if and only if all cardinals are nonmeasurable.

Example 1. Let K be a field carrying the trivial valuation and consider the power series field $F = K[[x]] = \left\{ \sum_{N}^{\infty} a_n x^n \,\middle|\, a_n \in K, N \in Z \right\}$ where addition in F is taken componentwise and multiplication is taken to be the Cauchy product. We define a valuation $|\ |$ on F by choosing a real number r such that $0 < r < 1$ and, assuming $a_N \neq 0$, we take $\left| \sum_{N}^{\infty} a_n x^n \right| = r^N$. Clearly the residue class field of F is K itself.

By [6, p. 41], there exist fields K of arbitrarily large cardinality.

THEOREM 1. *Let* F *be a complete discretely valued field whose residue class field has nonmeasurable cardinal and* μ *a regular monotone* σ*–smooth* 0 — 1 *measure on* $\mathcal{S}$. *Then* M^{μ} *is the kernel of a homomorphism of* C(T, F) *into* F.

Proof. Suppose $f \in C(T, F)$ and $f \notin M^{\mu}$. We wish to demonstrate the existence of a scalar $a \in F$ such that $z(f - ak_T) = f^{-1}(a) \in Z(M^{\mu})$. Let V

(*) Pervenuta all'Accademia il 5 luglio 1972.

(**) Subsequent investigations (*Repletions of Ultraregular spaces*, by the authors and S. Warner, to appear) have shown that some of the results obtained here are true in more general cases. In Theorems 1–3 and Prop. 15, for example, " discretely valued " can be removed.

be the valuation ring of F and P the maximal ideal of nonunits. We may write $V = \bigcup_S (a_r + P)$ where the scalars a_r come from a set of representatives S for the distinct cosets of the residue class field V/P. Choosing $\pi \in F$ such that $|\pi| < 1$ is a generator of the value group of F, we see that $F = \bigcup_{i=-\infty}^{\infty} \pi^i V$ and $T = \bigcup_{i=-\infty}^{\infty} f^{-1}(\pi^i V)$. Thus as μ is σ-smooth, there exists i_0 such that $\mu(f^{-1}(\pi^{i_0} V)) = 1$.

We note that $\pi^{i_0} V = \bigcup_{a_r \in S} (\pi^{i_0} a_r + \pi^{i_0} P)$. We also observe that for any set $L \subset S$, the set $A = \bigcup_{a_r \in L} (\pi^{i_0} a_r + \pi^{i_0} P)$ is a clopen subset of F and therefore $\mu(f^{-1}(A))$ may be considered. Defining $\hat{\mu}(L) = \mu(f^{-1}(A))$, we see that we have defined a σ-smooth monotone $0 - 1$ measure on the collection of all subsets of S. Since we have assumed that the residue class field has nonmeasurable cardinal, there exists $a_{r_0} \in S$ such that $\hat{\mu}(a_{r_0}) = \mu(f^{-1}(\pi^{i_0} a_{r_0} + \pi^{i_0} P)) = 1$. We repeat the same argument using the set $(\pi^{i_0} a_{r_0} + \pi^{i_0} P)$ in place of $\pi^{i_0} V$ and we find a_{r_1} such that $\mu(f^{-1}(\pi^{i_0} a_{r_0} + \pi^{i_0+1} a_{r_1} + \pi^{i_0+1} P)) = 1$. Continuing in this way we construct a nest A_n of clopen subsets of the field F whose diameters tend towards 0 and such that $\mu(f^{-1}(A_n)) = 1$. As F is complete, $\bigcap_{n=1}^{\infty} A_n = \{a\}$. As μ is σ-smooth, $\mu(f^{-1}(a)) = 1$.

COROLLARY 1. *If* F *is a complete discretely valued field, whose residue class field has nonmeasurable cardinal, then there is a* $1 - 1$ *correspondence between the regular monotone* σ*-smooth measures on* $\mathfrak{S}$ *and the homomorphisms of* C(T, F) *into* F.

COROLLARY 2. *With hypothesis as in Corollary* 1, *a maximal ideal* M^{μ} *of* C(T, F) *is the kernel of a homomorphism if and only if* $Z(M^{\mu})$ *is closed under the formation of denumerable intersections.*

PROPOSITION 10. *If* β *is a nonmeasurable cardinal,* $\beta^{\aleph_0}$ *is a nonmeasurable cardinal.*

Proof. Let F be a field such that the cardinality of the residue class field of F is greater than or equal to β and F is discretely valued. Endow F with discrete topology and denote F with this topology as T. Let μ be a σ-smooth monotone $0 - 1$ measure on $\mathfrak{S}$. By the result of Theorem 1, for every $f \in C(T, F)$ there exists a scalar $a \in F$ such that $\mu(z(f - ak_T)) = 1$. If we choose f to be a bijection between T and F we conclude that there is a point $a \in F$ such that $\mu(a) = 1$. Thus the cardinal associated with $T = F$ is nonmeasurable and so is $\beta^{\aleph_0}$.

COROLLARY. *The cardinal* $c = 2^{\aleph_0}$ *is nonmeasurable.*

THEOREM 2. *Let* F *be any complete discretely valued field. The nontrivial homomorphisms of* C(T, F) *are in* $1 - 1$ *correspondence with the regular* *monotone σ–smooth* $0 - 1$ *measures on* $\mathfrak{B}$ *if and only if all cardinals are nonmeasurable.*

Proof. Let β be an arbitrary cardinal and F a complete discretely valued field whose residue class field has cardinality greater than or egual to β. Assuming the homomorphisms of $C(T, F)$ into F are in $1 - 1$ correspondence with the regular monotone σ–smooth $0 - 1$ measures for any space T, we take $T = F$ endowed with discrete topology and by an argument similar to that of Proposition 10 we conclude that the σ–smooth measures on T are concentrated at points. Thus the cardinal of V/P and therefore β is nonmeasurable.

The converse is trivial.

Example 2. Let F be any complete nonarchimedean nontrivially valued field. Let X be a subalgebra of $C(T, F)$ which is " closed under inverses " (if $x \in X$ and $x^{-1} \in C(T, F)$, then $x^{-1} \in X$). We apply Michael's proof [4, p. 51] and observe that if conditions (1) and (2) below are satisfied, then the homomorphisms of X into F are generated by the points of T.

(1) For any n elements $x_1, \cdots, x_n \in X$ such that $\bigcap_{i=1}^{n} z(x_i) = \varnothing$, there exists $y_1, \cdots, y_m \in X$ such that $\sum_{i=1}^{n} x_i y_i = k_T$.

(2) There exists $x_1, \cdots, x_m \in X$ such that for any $\alpha_1, \cdots, \alpha_m \in F$ $\bigcap_{i=1}^{m} z(x_i - \alpha_i k_T)$ is compact.

We note that if $X = C(T, F)$, then X satisfies (1) (Prop. 4). We observe that if there exists a bijection $x \in C(T, F)$, then $C(T, F)$ will satisfy (2). Thus if we choose $T = F$ and endow T with any topology finer than the topology induced by the valuation (with the restriction that T remains a 0–dimensional Hausdorff space), then $C(T, F)$ will satisfy both (1) and (2). Thus, in particular, any complete nonarchimedean nontrivially valued field F is an $F - Q$ space [1].

We may now reduce the question of the existence of nonmeasurable cardinals to the following statement.

Every cardinal is nonmeasurable if and only if each maximal ideal $M \subset C(T, F)$ such that $Z(M)$ is closed with respect to the formation of denumerable intersections, is the kernel of a homomorphism of $C(T, F)$ into F where F is any complete discretely valued field and $T = F$ endowed with discrete topology.

THEOREM 3. *If* F *is a complete discretely valued field whose residue class field has nonmeasurable cardinal, then the regular monotone σ–smooth* $0 - 1$ *measures on the Baire sets of* F *are concentrated at points.*

Proof. We note that the homomorphisms of $C(F, F)$ are concentrated at points by the previous example while the σ–smooth measures are in $1 - 1$ correspondence with the homomorphisms by (Theorem 1).

4. THE NONARCHIMEDEAN STONE–ČECH COMPACTIFICATION OF T

In this section we approach an analog of the classical Stone–Čech compactification from a measure theoretic point of view. We denote by $M_0(T)$ the collection of all monotone 0 — 1 measures on $\mathcal{S}$, $M'_\sigma(T)$ the σ–smooth measures in $M_0(T)$, and $M_p(T)$ the measures concentrated at points of T. With an appropriate topology on $M_0(T)$ we show that $M_p(T)$ is homeomorphic to $T(M_p(T)$ considered as a subspace of $M_0(T))$, that $M_0(T)$ is a Hausdorff 0–dimensional compactification of T, and that any bounded continuous function taking $M_p(T)$ into a local field can be extended to a continuous function taking $M_0(T)$ into that field. Thus $M_0(T)$ is the nonarchimedean analog of the Stone–Čech compactification of T. (Moreover $M_0(T)$ is the Banaschewski compactification of T.)

DEFINITION 5. Let $X = \{0, 1\}^{\mathcal{S}}$ where $\mathcal{S} = \{S_\alpha \mid \alpha \in \Lambda\}$. A typical element of X is denoted by (a_α).

If X carries the product topology, a net $((a^\gamma_\alpha)) \to ((a_\alpha))$ if and only if for any index α_0 there exists γ_0 such that for all $\gamma \geq \gamma_0$, $a^\gamma_{\alpha_0} = a_{\alpha_0}$.

DEFINITION 6. Consider the subsets of $M_0(T)$ defined by

$$V(\mu_0 ; S_{\alpha_1}, \cdots, S_{\alpha_n}) = \{\mu \in M_0(T) \mid \mu(S_{\alpha_i}) = \mu_0(S_{\alpha_i}), i = 1, \cdots, n\}.$$

These sets form a base for a topology on $M_0(T)$ which is referred to as *the weak clopen topology*.

DEFINITION 7. The injective mappings φ and ψ are defined as follows

$$\varphi : T \to M_0(T)$$

$$t \to \mu_t$$

$$\psi : M_0(T) \to X$$

$$\mu \to (\mu(S_\alpha))$$

where μ_t denotes the measure concentrated at $t \in T$.

We should like to consider X with a stronger topology than the product topology. We refer to the topology generated by neighborhoods of (a_α) of the form

$$V((a_\alpha) ; \alpha_1, \cdots, \alpha_n, \cdots) = \{(b_\alpha) \mid b_{\alpha_i} = a_{\alpha_i}, \ i = 1, 2 \cdots\}$$

as the *denumerable box topology*. With subsets of $X \times X$ of the form

$$\{((a_\alpha), (b_\alpha)) \mid a_{\alpha_i} = b_{\alpha_i}, \ i = 1, 2, \cdots\}$$

considered, we see that these are entourages for a uniform structure generating the denumerable box topology and that X is a complete uniform space with respect to this uniform structure. Carrying both the uniform structure and the denumerable box topology back to $M_0(T)$ through the map ψ in the usual way, we say that $M_0(T)$ carries the denumerable box topology as well. We see that under these circumstances, the mapping ψ is uniformly continuous.

PROPOSITION 10. (a) *If* T *carries its original topology and* $M_0(T)$ *the weak clopen topology, then* φ *is a homeomorphism.*

(b) *If* $M_0(T)$ *carries the weak clopen topology, and* X *carries the product topology, then* ψ *is a homeomorphism.*

Proof. (*a*) We note that if $t \in S_\alpha$, then a net (t_γ) converges to $t \in T$ if and only if the net is eventually in S_α. If $t \notin S_\alpha$, then (t_γ) converges to t if and only if (t_γ) is eventually in CS_α. In either case we see that $\mu_{t_\gamma}(S_\alpha)$ is eventually equal to $\mu_t(S_\alpha)$ and therefore $\mu_{t_\gamma} \to \mu_t$. Hence the map φ is continuous. The continuity of φ^{-1} follows in the same way.

(*b*) The bicontinuity of ψ is clearly a consequence of the definitions of the various topologies.

PROPOSITION 11. *The closure of* $\psi(M_p(T))$ *in* X (*endowed with product topology*) *is* $\psi(M_0(T))$.

Proof. We begin by showing that $M_p(T)$ is dense in $M_0(T)$ (weak clopen topology) so that clearly it will follow that $\psi(M_p(T))$ is dense in $\psi(M_0(T))$. We take any $\mu \in M_0(T)$ and let H denote the indexing set for the clopen sets S_α such that $\mu(S_\alpha) = 1$. We order H by $\alpha_1 \leqq \alpha_2$ if and only if $S_{\alpha_2} \subset S_{\alpha_1}$. Since $\mu(S_{\alpha_1} \cap S_{\alpha_2}) = 1$, we see that H is a directed set. For each $\alpha \in H$ we choose $t_\alpha \in S_\alpha$ and contend that the net $\mu_{t_\alpha} \to \mu$.

If $\mu(S) = 1$, then $S = S_\alpha$ for some α and therefore by the preceeding, it follows that $\mu_{t_\alpha}(S)$ is eventually equal to 1. A similar argument holds if $\mu(S) = 0$.

To show that $\overline{\psi M_p(T)}$ is $\psi M_0(T)$ we need only show that $\psi M_0(T)$ is closed in X. Let (a_α) be the limit in the product topology of a net $((\mu_\gamma(S_\alpha)))$ where $\mu_\gamma \in M_0(T)$ for every γ. We define $\mu(S_\alpha) = a_\alpha$ and observe that μ is readily shown to be a monotone $0-1$ measure on $\mathcal{S}$.

PROPOSITION 12. *Let* T *carry its original topology and* $M_p(T)$ *the denumerable box topology. The mapping* ψ *is a homeomorphism if and only if all* C_δ *sets are open in* T.

Proof.

$$\varphi^{-1}\{\mu_t \in M_p(T) \mid \mu_t(S_{\alpha_i}) = \mu_{t_0}(S_{\alpha_i})\ i = 1, 2, \cdots\} = \left(\bigcap_{t_0 \in S_{\alpha_i}} S_{\alpha_i}\right) \cap \left(\bigcap_{t_0 \notin S_{\alpha_i}} CS_{\alpha_i}\right).$$

Clearly these sets are open in T if and only if all C_δ sets are open.

PROPOSITION 13. *The completion of* $M_p(T)$ *when* $M_p(T)$ *carries the denumerable box topology and its associated uniform structure is* $M_\sigma(T)$.

Proof. We show that the closure of $\psi M_p(T)$ is $\psi M_\sigma(T)$ in the complete space X (carrying the denumerable box topology).

To prove that $\psi M_p(T)$ is dense in $\psi M_\sigma(T)$ we let μ denote any monotone σ–smooth $0 - 1$ measure on $\mathcal{S}$ and use the fact that μ can be extended to the Baire sets $\mathcal{B}$. We consider the collection of C_δ sets

$$\left\{ H_\beta = \bigcap_{i=1}^{\infty} S_i^\beta \mid \hat{\mu}(H_\beta) = 1 \right\}.$$

Once again as in (Proposition 11) we show that the indexing set for this collection is a directed set and choosing $t_\beta \in H_\beta$ we can prove in a similar way that $\mu_{t_\beta} \to \mu$ in the denumerable box topology.

To show that $\psi M_\sigma(T)$ is closed in X we take a net $(\mu_\gamma(S_\alpha)) \to (a_\alpha)$, define $\mu(S_\alpha) = a_\alpha$, and observe that μ is a monotone $0 - 1$ σ–smooth measure on $\mathcal{S}$.

COROLLARY 1. $M_p(T) = M_\sigma(T)$ *if and only if* $M_p(T)$ *is a complete uniform space when it carries the denumerable box topology and its associated uniform structure.*

THEOREM 4. *Let* F *be a local field and suppose* $M_0(T)$ *carries the weak clopen topology. Then a bounded continuous function* $f: M_p(T) \to F$ *can be extended to a continuous function* $\hat{f}: M_0(T) \to F$ (*).

Proof. Let (μ_{t_γ}) be a net from $M_p(T)$ converging to $\mu \in M_0(T)$. As $(f(\mu_{t_\gamma}))$ is a bounded net of points in the local field F, there must be a subnet $f(\mu_{t_{\gamma(k)}}) \to a$ for some $a \in F$. Letting $\hat{f}(\mu) = a$, we should like to show that $\hat{f}$ is the desired extension of f.

To begin we show that $f(\mu_{t_\gamma}) \to \hat{f}(\mu)$. If not, then for some $\varepsilon > 0$ and $S_0 = f^{-1}(\{b \in F \mid |b - a| < \varepsilon\})$, μ_{t_γ} is not eventually in S_0. It then follows that $\mu(CS_0) = 1$ or equivalently that μ_{t_γ} is eventually in CS_0. But this contradicts the fact that the subnet $f(\mu_{t_{\gamma(k)}})$ converges to a.

We show that $\hat{f}(\mu) = a$ is well defined. To do this suppose that another net $\mu_{t_\delta} \to \mu$ and $f(\mu_{t_\delta}) \to b$. If we suppose that $a \neq b$, then taking $\varepsilon > 0$ such that $|a - b| > \varepsilon$, we see that μ_{t_γ} is eventually in $S_0 = f^{-1}(\{c \in F \mid |c - a| < \varepsilon\})$, while μ_{t_δ} is eventually in CS_0. In order for both nets to converge to μ, it must follow that $\mu_{t_\gamma}(S_0)$ and $\mu_{t_\delta}(S_0)$ are both eventually equal to $\mu(S_0)$. But this cannot be true as a consequence of the preceeding conclusions. Thus it follows that $a = b$ and $\hat{f}$ is well defined.

(*) This argument holds for arbitrary fields F and function $f \in C(T, F)$ such that the range of f is relatively compact.

To show that f is continuous suppose $\mu_\gamma \to \mu$ and $\hat{f}(\mu_\gamma) \not\to \hat{f}(\mu)$. Then for some $\varepsilon > 0$, the net $(\hat{f}(\mu_\gamma))$ is not eventually in the set $\{a \in F \,|\, |a - \hat{f}(\mu)| < \varepsilon\}$. As F is a local field and the function f is bounded, we can construct a finite partition $S_{i,\varepsilon}$ of clopen subsets of $M_p(T)$ such that the diameter of $f(S_{i,\varepsilon})$ is less than ε. As $M_p(T) = \bigcup_{i=1}^{n} S_{i,\varepsilon}$, it is clear that for some i_0, $\mu(S_{i_0,\varepsilon}) = 1$, while for all other i, $\mu(S_{i,\varepsilon}) = 0$. (Here, for the moment, we are identifying the spaces $M_p(T)$ and T). Thus, for sufficiently large γ, $\mu_\gamma(S_{i,\varepsilon}) = 0$ for $i \neq i_0$, while $\mu_\gamma(S_{i_0,\varepsilon}) = 1$. For each γ we may construct a net $\mu_{t_{\beta(\gamma)}} \to \mu_\gamma$ as well as a net $\mu_{t_\beta} \to \mu$. It is clear that each of these nets must ultimately be in $S_{i_0,\varepsilon}$ if, at this point, we make the assumption that γ is so large that $\mu_\gamma(S_{i_0,\varepsilon}) = 1$. Since $\hat{f}(\mu_{t_{\beta(\gamma)}}) \to \hat{f}(\mu_\gamma)$ and $\hat{f}(\mu_{t_\beta}) \to \hat{f}(\mu)$, the facts that $\mu_{t_{\beta(\gamma)}}$ and μ_{t_β} are eventually located in $S_{i_0,\varepsilon}$ and the diameter of $f(S_{i_0,\varepsilon})$ is less than ε, preclude the possibility that $|\hat{f}(\mu_\gamma) - \hat{f}(\mu)|$ is frequently greater than ε. Thus it is clear that $\hat{f}(\mu_\gamma) \to \hat{f}(\mu)$.

Thus we see that $M_0(T)$ with the weak clopen topology is the nonarchimedean analog of the Stone–Čech compactification of T. We refer to this space as $\beta_0(T)$.

The classical Gelfand–Kolmogoroff theorem [2, p. 210] is proved for the case in which the field of coefficients is the real numbers. We now show that an analog of this theorem is true for all complete nonarchimedean valued fields.

THEOREM 5 (Gelfand–Kolmogoroff). *The maximal ideal* M^μ *of* $C(T, F)$ *satisfies the relationship*

$$M^\mu = \{f \in C(T, F) \,|\, \mu \in \mathrm{cl}_{\beta_0(T)}\, z(f)\}.$$

This establishes a $1 - 1$ *correspondence between the points of* $\beta_0(T)$ *and the maximal ideals of* $C(T, F)$.

Proof. Under the correspondence of Proposition 6, there is a $1 - 1$ correspondence between the points of $\beta_0(T)$ and the maximal ideals of $C(T, F)$ where the relationship

$$M^\mu = \{f \in C(T, F) \,|\, \hat{\mu}(z(f)) = 1\}.$$

Identifying, for the purposes of this proof, the spaces T and $M_p(T)$ we wish to show that $\mu \in \mathrm{cl}_{\beta_0(T)}\, z(f)$ for each $f \in M^\mu$. Once again we note that $H = \{\alpha \in \Lambda \,|\, \mu(S_\alpha) = 1\}$ is a directed set and since $\mu(S_\alpha \cap z(f)) = 1$ for each $\alpha \in H$, we may choose $t_\alpha \in z(f) \cap S_\alpha$ for each $\alpha \in H$. We show that $\mu_{t_\alpha} \to \mu$ as follows. Consider any neighborhood $V(\mu; S_{\alpha_1}, \cdots, S_{\alpha_n})$ of μ in the vague topology. Suppose that $\mu(S_{\alpha_i}) = 1$ for $i = 1, \cdots, j$ while $\mu(S_{\alpha_i}) = 0$ for $i > j$. If $1 \leq i \leq j$, $\alpha_i \in H$ and therefore $t_\alpha \in S_{\alpha_i}$ for sufficiently large indices α and all i such that $1 \leq i \leq j$. If $i > j$ $\mu(CS_{\alpha_i}) = 1$ and therefore for sufficiently large $\alpha \in H$ $t_\alpha \notin S_{\alpha_i}$ for all $i > j$. Thus, ultimately, $\mu(S_{\alpha_i}) = \mu_{t_\alpha}(S_{\alpha_i})$

for all i such that $1 \leqq i \leqq n$ and therefore for sufficiently large α, $\mu_{t_\alpha} \in V(\mu; S_{\alpha_1}, \cdots, S_{\alpha_n})$.

We now show that if $\mu \in \mathrm{cl}_{\beta_0(T)} z(f)$ for some $f \in C(T, F)$, then $f \in M^\mu$. To do this we need only show that $\hat{\mu}(z(f)) = 1$. If we assume that $\hat{\mu}(z(f)) = 0$, then there exists $g \in C(T, F)$ such that $z(f) \cap z(g) = \varnothing$ and $\hat{\mu}(z(g)) = 1$. By Proposition 3 there exists a clopen set $S \in \mathcal{S}$ such that $z(g) \subset S$ while $z(f) \subset CS$. It then follows that $z(f) \cap V(\mu; S) = \varnothing$ and $\mu \notin \mathrm{cl}_{\beta_0(T)} z(f)$.

5. $\hat{Q}$–SPACES

In this section we consider a nonarchimedean analog of realcompact spaces (Q–spaces) and realcompactifications of a space.

DEFINITION 8. T is a Q_0–space, if and only if $M_\sigma(T) = M_p(T)$. That is, T is a Q_0–space if and only if the monotone $0 - 1$ σ–smooth measures are concentrated at points of T.

By preceeding results (Proposition 8, Theorems 1 and 3) the following statements are all true.

PROPOSITION 14. *T is a Q_0–space if and only if the z–ultrafilters Z closed with respect to the formation of denumerable intersections, are fixed* ($\cap Z \neq \varnothing$).

PROPOSITION 15. *Let* F *be a discretely valued field whose residue class field has nonmeasurable cardinal. Then* T *is a Q_0–space if and only if the nontrivial homomorphisms of* $C(T, F)$ *into* F *are evaluation maps.*

PROPOSITION 16. *If* F *is a discretely valued field whose residue class field has nonmeasurable cardinal, then* F *is a Q_0–space.*

PROPOSITION 17. *A closed subspace of a Q_0–space is a Q_0–space.*

Proof. Let T be a Q_0–space and $E \subset T$ be a closed subspace of T. Let μ be a σ–smooth measure on the clopen subsets of E. Induce a measure $\hat{\mu}$ on the clopen subsets of T by defining $\hat{\mu}(S) = \mu(S \cap E)$ for any clopen set $S \subset T$. It is clear that $\hat{\mu}$ is a σ–smooth monotone $0 - 1$ measure on T and as T is a Q_0–space, $\hat{\mu}$ is concentrated at $t \in T$. We contend that $t \in E$. Otherwise there exists a clopen set $S \subset T$ such that $t \in S$ while $S \cap E = \varnothing$. However, it follows that $\mu(S) = 1 = \mu(S \cap E) = \mu(\varnothing)$ which is a contradiction. It follows from the 0–dimensionality of T and the fact that E is closed, that if $H \subset E$, $\mu(H) = 1$, and H is clopen in E, then $t \in H$.

PROPOSITION 18. *A product of Q_0–spaces is a Q_0–space.*

Proof. Let $T = \Pi T_\alpha$ where each T_α is a Q_0–space. Let p_α denote the projection of T onto T_α. Let μ be a monotone σ–smooth $0 - 1$ measure on the clopen subsets of T. Let μ_α be the monotone σ–smooth $0 - 1$ measure on the clopen subsets of T_α induced by the relationship $\mu_\alpha(S_\alpha) = \mu(p_\alpha^{-1}(S_\alpha))$

where S_α is a clopen subset of T_α. As T_α is a Q_0–space, μ_α is concentrated at some point $t_\alpha \in T$. We show that μ_α is concentrated at (t_α).

Let 0 be a clopen subset of T and suppose $(t_\alpha) \in 0$. Then $(t_\alpha) \in S_{\alpha_1} \times \times S_{\alpha_2} \times \cdots \times S_{\alpha_n} \times \prod_{\alpha \neq \alpha_i} T_\alpha \subset 0$. As $t_{\alpha_i} \in S_{\alpha_i}$ for $1 \leqq i \leqq n$, $\mu_{\alpha_i}(S_{\alpha_i}) = 1$ for $1 \leqq i \leqq n$. Thus $\mu(p_{\alpha_i}^{-1}(S_{\alpha_i})) = 1$ for $1 \leqq i \leqq n$ and as a finite intersection of sets of measure 1 has measure 1, it follows that $\mu(0) = 1$ and therefore that μ is concentrated at (t_α).

PROPOSITION 19. *Let* $T = \prod_\alpha T_\alpha$ *where each* T_α *is a* Q_0*–space. Then* T *endowed with the "denumerable box topology" is a* Q_0*–space.*

Proof. The argument of the previous proposition holds without essential change, except for using the fact that a denumerable intersection of sets having μ–measure equal to 1, will have measure equal to 1.

COROLLARY. $M_\sigma(T)$ *with denumerable box topology is a* Q_0*–space.*

Proof. $M_\sigma(T)$ is homeomorphic to a closed subspace of the Q_0–space $X = \{0, 1\}^{\mathcal{S}}$ (carrying the denumerable box topology).

THEOREM 6. *Let* $\beta_0(T)$ *carry the weak clopen topology. Let* F *be a local field. Then every* $f \in C(T, F)$ *can be extended to* $\hat{f} \in C(M_\sigma(T), F)$. *If* $\mu \notin M_\sigma(T)$, *then there exists* $\hat{f} \in C(M_\sigma(T), F)$ *which cannot be continuously extended to* μ.

Proof. Let (t_γ) be a net such that $t_\gamma \to \mu \in M_\sigma(T)$. If we could show that for any $f \in C(T, F)$, the net $(f(t_\gamma))$ is ultimately bounded, then we could apply the argument of Theorem 4 to show that f can be continuously extended.

To demonstrate this, we observe that if $S_n = \{t \in T \,|\, |f(t)| \leqq n\}$ where n is any positive integer, there must be some n such that $S_n \in Z(M^\mu)$. Otherwise, if $CS_n \in Z(M^\mu)$ for all n, then the z–ultrafilter $Z(M^\mu)$ is not closed with respect to the formation of denumerable intersections as $\bigcap_{n=1}^{\infty} CS_n = \varnothing$. Thus, for some n, $\mu(S_n) = 1$ and as $t_\gamma \to \mu$, it follows that (t_γ) is ultimately in S_n. The net $(f(t_\gamma))$ is therefore ultimately bounded.

We wish to show that if $\mu \notin M_\sigma(T)$, then there exists $f \in C(T, F)$ such that if a net (t_γ) of points in T converges to μ, then $|f(t_\gamma)| \to \infty$. As $\mu \notin M_\sigma(T)$, there exists a descending sequence (S_n) of clopen sets such that $S_n \in Z(M^\mu)$ for all positive integers n while $\bigcap_{n=1}^{\infty} S_n = \varnothing$. Let $\alpha \in F$ be chosen such that $|\alpha| > 1$. The function $f = \sum_{n=1}^{\infty} \alpha^n k_{CS_n \cap S_{n-1}}$ will have the desired properties.

THEOREM 7. $M_\sigma(T)$ *with the weak clopen topology is a* Q_0*–space.*

Proof. We wish to show that $M_\sigma M_\sigma(T) = M_\sigma(T)$. We observe that $\beta_0(T) = \beta_0(M_\sigma(T))$ and by Theorem 6 the algebra $C(T, F)$ can be identified with $C(M_\sigma(T), F)$ in the case where F is a local field. As there is a function $\hat{f} \in C(M_\sigma(T), F)$ which cannot be continuously extended to μ for any $\mu \notin M_\sigma(T)$, it follows by the previous theorem that $M_\sigma(M_\sigma(T)) = M_\sigma(T)$.

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