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Comparison theorems for fourth order quasilinear matrix differential inequalities

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Matematica. — *Comparison theorems for fourth order quasilinear matrix differential inequalities.* Nota di C. Y. Chan e E. C. Young, presentata (*) dal Socio M. Picone.

Riassunto. — Nel presente articolo si stabilisce una « identità Picone » per una classe di matrici di quasilineari diseguaglianze differenziali del quart'ordine. La citata identità servirà a mostrare teoremi di confronto del tipo di Sturm.

1. Introduction

Picone integral identity [1, p. 266] for second order ordinary differential equations is used mainly in establishing a more general comparison theorem than that originally given by Sturm [1, pp. 224–226]. It was extended to second order elliptic differential equations by Picone [2], Kreith [3–4], Dunninger [5], and Dunninger and Weinacht [6], to self-adjoint strongly elliptic systems by Kreith [7–8], and Kreith and Travis [9], and to fourth order elliptic equations by Dunninger [10]. Sturm comparison theorems for second order self-adjoint elliptic systems were given by Kuks [11], Bochenek [12], Diaz and McLaughlin [13], Kreith [7–8], Kreith and Travis [9], and Swanson [14] while Noussair [15] extended the result of Swanson [16] for quasilinear self-adjoint systems to the nonself-adjoint case. Higher order elliptic equations and inequalities were considered by Dunninger [10], and Diaz and Dunninger [17–18].

The purpose of this paper is to extend Picone identity and Sturm comparison theorems to a class of fourth order quasilinear elliptic systems. We establish Picone integral identity in section 2, and then prove Sturm comparison theorems for self-adjoint as well as nonself-adjoint systems in section 3.

2. Picone integral identity

Let Ω be a nonempty bounded domain in the n–dimensional Euclidean space E^n, $\overline{\Omega}$ its closure and $\partial\Omega$ its boundary. Let x denote a point $(x_1, x_2, \cdots, x_n)$ of E^n, D_i differentiation with respect to x_i, and Δ the Laplace operator in the variables $x_1, x_2, \cdots, x_n$.

We consider the matrix differential system

$$(2.1) \qquad LV \equiv \Delta [A(x, V) \Delta V] + 2 B(x, V) \Delta V + P(x, V) V$$

(*) Nella seduta dell'11 novembre 1972.

for x in Ω, and V with range in H^m where H is a domain in E^m containing the origin. The coefficients A, B and P are real $m \times m$ matrix functions such that A belongs to $C^2(\overline{\Omega} \times H^m)$, and both B and P belong to $C(\overline{\Omega} \times H^m)$. We also consider the system

$$lu \equiv \Delta[a(x, u)\Delta u] + 2b(x, u)\Delta u + p(x, u)u \tag{2.2}$$

for a $m \times 1$ vector function u with range in H. The coefficients a, b and p are real $m \times m$ matrix functions such that a belongs to $C^2(\overline{\Omega} \times H)$ and both b and c belong to $C(\overline{\Omega} \times H)$. The domain D of l is the set of all real-valued functions of class $C^4(\Omega) \cap C^2(\overline{\Omega})$ with range in H. The set of all $m \times m$ matrix functions whose column vectors $V_i \in D$ $(i = 1, 2, \cdots, m)$ is denoted by D^m.

In analogy to the case of second order elliptic systems (cf. Kreith and Travis [9], and Swanson [14]), we say that a matrix V is L-prepared if

$$V^T A\Delta V = (\Delta V^T) AV, \tag{2.3}$$

$$(D_i V^T) A\Delta V = (\Delta V^T) AD_i V \qquad \text{for} \quad i = 1, 2, \cdots, n, \tag{2.4}$$

where V^T denotes the transpose of V. Conditions (2.3) and (2.4) are obviously true in the scalar case when $m = 1$. When $m = 2$, $n = 1$, and A is the identity matrix,

$$V = \begin{pmatrix} e^x & e^{-x} \\ e^{-x} & e^x \end{pmatrix}$$

for example is a L-prepared matrix.

We use the convention that whenever the index i is repeated in a term, such a term is to be summed from 1 to n. Let V^{-1} denote the inverse of V, and G be an arbitrary $m \times m$ matrix. Let us define

$$f[u] \equiv \int_\Omega [(\Delta u)^T a\Delta u + 2u^T b\Delta u + u^T pu]\,dx,$$

$$F[u, V] \equiv \int_\Omega [(\Delta u)^T A\Delta u + 2u^T B\Delta u + u^T (P + G)u]\,dx,$$

$$J[u, V] \equiv -2\int_\Omega [D_i u - (D_i V)V^{-1}u]^T A(\Delta V)V^{-1}[D_i u - (D_i V)V^{-1}u]\,dx,$$

$$Q[u, V] \equiv \int_\Omega \{[\Delta u - (\Delta V)V^{-1}u]^T A[\Delta u - (\Delta V)V^{-1}u]$$

$$+ 2u^T B[\Delta u - (\Delta V)V^{-1}u] + u^T Gu\}\,dx.$$

THEOREM 1. *If* $\partial\Omega$ *is piecewise smooth and* $u \in D$, *then every* L*–prepared matrix* $V \in D^m$ *such that* $V^{-1} u \in C^2(\overline{\Omega})$ *with range in* H *satisfies*

$$(2.5)\qquad \int_\Omega [u^T lu - u^T (LV) V^{-1} u]\, dx = f[u] - F[u, V] + J[u, V] + Q[u, V]$$

$$+ \int_{\partial\Omega} \left[u^T \frac{\partial (a\Delta u)}{\partial n} - \frac{\partial u^T}{\partial n} a\Delta u - u^T \frac{\partial (A\Delta V)}{\partial n} V^{-1} u \right.$$

$$\left. + \frac{\partial u^T}{\partial n} (A\Delta V) V^{-1} u + u^T A (\Delta V) \frac{\partial (V^{-1} u)}{\partial n} \right] ds,$$

where $\partial/\partial n$ *denotes the outward normal derivative.*

Proof. By direct computation,

$$(2.6)\qquad u^T \Delta (A\Delta V) V^{-1} u$$

$$= D_i [u^T D_i (A\Delta V) V^{-1} u - (D_i u^T) A (\Delta V) V^{-1} u - u^T A (\Delta V) D_i (V^{-1} u)]$$

$$+ (\Delta u^T) A (\Delta V) V^{-1} u + 2 (D_i u^T) A (\Delta V) D_i (V^{-1} u) + u^T A (\Delta V) \Delta (V^{-1} u).$$

Using the relation $D_i V^{-1} = - V^{-1} (D_i V) V^{-1}$, we have

$$(2.7)\qquad D_i (V^{-1} u) = - V^{-1} (D_i V) V^{-1} u + V^{-1} D_i u,$$

$$(2.8)\qquad \Delta (V^{-1} u) = 2 V^{-1} (D_i V) V^{-1} (D_i V) V^{-1} u - V^{-1} (\Delta V) V^{-1} u$$

$$- 2 V^{-1} (D_i V) V^{-1} D_i u + V^{-1} \Delta u.$$

From (2.3), we obtain

$$(2.9)\qquad A (\Delta V) V^{-1} = (V^{-1})^T (\Delta V^T) A.$$

By using (2.4), (2.7), (2.8) and (2.9), the last three terms on the right hand side of (2.6) can be written as

$$(2.10)\qquad 2 [D_i u - (D_i V) V^{-1} u]^T A (\Delta V) V^{-1} [D_i u - (D_i V) V^{-1} u]$$

$$- [\Delta u - (\Delta V) V^{-1} u]^T A [\Delta u - (\Delta V) V^{-1} u] + (\Delta u)^T A \Delta u.$$

By Green's second identity, we have

$$(2.11)\qquad \int_\Omega u^T \Delta (a\Delta u)\, dx = \int_\Omega (\Delta u)^T a\Delta u\, dx + \int_{\partial\Omega} \left[u^T \frac{\partial (a\Delta u)}{\partial n} - \frac{\partial u^T}{\partial n} a\Delta u \right] ds.$$

Picone integral identity (2.5) then follows from (2.1), (2.2), (2.6), (2.10) and (2.11) with the use of the divergence theorem.

We observe that if $a = b = p = 0$, then (2.5) reduces to the following form.

COROLLARY 1. *If* $\partial\Omega$ *is piecewise smooth, and* $u \in C^2(\overline{\Omega})$ *with range in* H, *then every* L*-prepared matrix* $V \in D^m$ *such that* $V^{-1}u \in C^2(\overline{\Omega})$ *with range in* H *satisfies*

$$(2.12) \qquad -\int_\Omega u^T (LV)\, V^{-1} u \, dx = -F[u, V] + J[u, V] + Q[u, V]$$

$$-\int_{\partial\Omega} \left[u^T \frac{\partial (A\Delta V)}{\partial n} V^{-1} u - \frac{\partial u^T}{\partial n} A(\Delta V) V^{-1} u - u^T A(\Delta V) \frac{\partial (V^{-1} u)}{\partial n} \right] ds .$$

3. STURM THEOREMS

Let (A) denote the $mn \times mn$ matrix whose (i, j)-th block is A for $i, j = 1, 2, \cdots, n$. Also let (B) denote the $mn \times m$ matrix whose $(i, 1)$-th block is B for $i = 1, 2, \cdots, n$. In this section, we assume that A is symmetric. If (A) is positive definite for all (x, ξ) in $\Omega \times H^m$, then it follows with slight modification from Gantmacher [19, p. 306] that a diagonal $m \times m$ matrix $G(x, \xi)$ in $C(\overline{\Omega})$ can be constructed such that for any $(x, \xi) \in \Omega \times H^m$ the matrix

$$(3.1) \qquad \begin{pmatrix} (A) & (B) \\ (B)^T & G \end{pmatrix} \text{ is positive definite in } \Omega \times H^m.$$

We now establish two weak Sturm theorems for L in the sense that the conclusion applies to $\overline{\Omega}$ rather than Ω.

THEOREM 2. *Let* $\partial\Omega$ *be piecewise smooth and* G *satisfy* (3.1). *If there exists a non-trivial* $u \in C^2(\overline{\Omega})$ *with range in* H *such that* $u = 0$ *on* $\partial\Omega$ *and* $F[u, V] \leqq 0$, *then every* L*-prepared matrix* $V \in D^m$ *for which* $V^T LV$ *and* $-V^T A\Delta V$ *are positive semidefinite in* Ω *must be singular at some point on* $\overline{\Omega}$.

Proof. We first remark that the condition that $-V^T A\Delta V$ is positive semidefinite corresponds to the self-adjoint scalar case (cf. Dunninger [10]) that A is positive in Ω, $v > 0$ for some $x \in \Omega$ and $\Delta v < 0$ in Ω. We now proceed to the proof of the theorem.

If V is non-singular on $\overline{\Omega}$, then there exists a unique w in $C^2(\overline{\Omega})$ such that $u = Vw$ on $\overline{\Omega}$. Thus $V^{-1} u = w$, and hence

$$u^T (LV)\, V^{-1} u = w^T V^T (LV)\, w,$$

$$J[u, V] = -2 \int_\Omega (D_i w)^T V^T A (\Delta V) D_i w \, dx.$$

Since $F[u, V] \leqq 0$, $V^T LV$ and $-V^T A\Delta V$ are positive semidefinite, and G satisfies (3.1), the left hand side of (2.12) is non-positive while the right hand side is non-negative. Hence each term of (2.12) must vanish. In particular, $Q[u, V] = 0$ in Ω. By (3.1), this implies $u \equiv 0$ in Ω, and hence on $\overline{\Omega}$ by continuity. Thus we have a contradiction to the fact that u is non-trivial. Therefore V must be singular at some point of $\overline{\Omega}$.

We have the following result when L is self-adjoint, that is $B \equiv 0$.

THEOREM 3. *Let* $\partial\Omega$ *be piecewise smooth and* (A) *be positive semidefinite for* (x, ξ) *in* $\Omega \times H^m$. *If there exists a non-trivial* $u \in C^2(\overline{\Omega})$ *with range in* H *such that* $u = 0$ *on* $\partial\Omega$ *and*

$$E[u, V] \equiv \int_\Omega [(\Delta u)^T A \Delta u + u^T Pu] \, dx \leqq 0,$$

then every L*–prepared matrix* $V \in D^m$ *for which* $V^T LV$ *is positive semidefinite and* $V^T A\Delta V$ *is negative definite in* Ω *must be singular at some point of* $\overline{\Omega}$.

Proof. If V is non-singular on $\overline{\Omega}$, then as in the proof of Theorem 2, we arrive at each term of (2.12), where B and G are taken to be identically zero, must vanish. In particular, $J[u, V] = 0$. Since $V^T A\Delta V$ is negative definite in Ω, this implies $D_i w = 0$ in Ω, and hence on $\overline{\Omega}$ by continuity. Therefore $u = Vr$ for some constant vector $r \neq 0$. This in turn implies that V is singular on $\partial\Omega$, and we have a contradiction.

The following two theorems give the strong versions of Theorems 2 and 3.

THEOREM 4. *Let* $\partial\Omega \in C^2$ *and* G *satisfy* (3.1). *If there exists a non-trivial* $u \in D$ *such that*

$$\int_\Omega u^T lu \, dx \leqq 0 \qquad \text{in} \quad \Omega, \tag{3.2}$$

$$u = 0 = \frac{\partial u}{\partial n} \qquad \text{on} \quad \partial\Omega, \tag{3.3}$$

$$f[u] \geqq F[u, V], \tag{3.4}$$

then every L*–prepared matrix* $V \in D^m$ *for which* $V^T LV$ *and* $-V^T A\Delta V$ *are positive semidefinite in* Ω *must be singular at some point in* Ω.

Proof. If V is non-singular in Ω, then there exists a unique $w \in C^4(\Omega)$ such that $u = Vw$. Since $\partial\Omega \in C^2$, $u \in C^2(\overline{\Omega})$ and $u = 0 = \frac{\partial u}{\partial n}$ on $\partial\Omega$, it follows from Agmon [20, p. 131] that u belongs to the Sobolev space $H_2^0(\Omega)$, the completion of the class of $C_0^\infty(\Omega)$ functions having compact support in Ω under the norm

$$\|u\|_2 = \left[\int_\Omega \left(|u|^2 + \sum_{j=1}^n \left|\frac{\partial^2 u}{\partial x_j^2}\right|^2\right) dx\right]^{1/2}. \tag{3.5}$$

Let $\{u_k\}$ be a sequence of $C_0^\infty(\Omega)$ functions converging to u in the norm (3.5). Since u_k vanishes in a neighbourhood of $\partial\Omega$, it follows that $V^{-1} u_k$ is at least of class $C^2(\overline{\Omega})$, and hence from (2.12), we have

$$F[u_k, V] \geqq Q[u_k, V] \geqq 0. \tag{3.6}$$

Since A, B, P and G are uniformly bounded for $x \in \overline{\Omega}$, there exists a positive constant M such that

$$|F[u_k, V] - F[u, V]| \leq M(n+1)^2 (\|u_k\|_2 + \|u\|_2) \|u_k - u\|_2 \tag{3.7}$$

with the use of Cauchy–Schwartz inequality. But $\|u_k - u\|_2 \to 0$ as $k \to \infty$; hence from (3.6) and (3.7), we have $F[u, V] \geq 0$. On the other hand, $f[u] \geq F[u, V]$ from (3.4), and $0 \geq f[u]$ from (2.11), (3.2) and (3.3). Thus we have a contradiction unless $F[u, V] = 0$.

If $F[u, V] = 0$, then let S be a ball such that $\overline{S} \subset \Omega$. Also let $u_k = V w_k$. Then $w_k = V^{-1} u_k$. Since $V^{-1} \in C^2(\overline{S})$, we have $\|w_k - w\|_{2,\omega} \to 0$ as $\|u_k - u\|_2 \to 0$, where the subscript ω indicates that the integral is evaluated over ω instead of over Ω. Now $w = V^{-1} u$, and

$$Q_\omega[u, V] = \int_\omega \{(\Delta u)^T A \Delta u - w^T (\Delta V)^T A \Delta u - (\Delta u)^T A (\Delta V) w$$

$$+ w^T (\Delta V)^T A (\Delta V) w + 2 u^T B [\Delta u - (\Delta V) w] + u^T G u\} \, dx.$$

Using Cauchy–Schwartz inequality and the uniform boundedness of A, B, G and ΔV, we can find a positive constant N such that

$$\begin{aligned} |Q_\omega[u_k, V] - Q_\omega[u, V]| \leq N \{&[(n+1)^2 (\|u_k\|_{2,\omega} + \|u\|_{2,\omega}) \\ &+ n \|w_k\|_{2,\omega} + (n+2) \|w\|_{2,\omega}] \, \|u_k - u\|_{2,\omega} \\ &+ [(n+2) \|u_k\|_{2,\omega} + n \|u\|_{2,\omega} \\ &+ \|w_k\|_{2,\omega} + \|w\|_{2,\omega}] \, \|w_k - w\|_{2,\omega}\}. \end{aligned}$$

Then as $k \to \infty$, $Q_\omega[u_k, V] \to Q_\omega[u, V]$, but $0 \leq Q_\omega[u_k, V] \leq Q[u_k, V]$. From (3.6) and (3.7), $0 \leq Q_\omega[u, V] \leq F[u, V]$. Since $F[u, V] = 0$, we have $Q_\omega[u, V] = 0$. This implies $u \equiv 0$ in S. Since S is arbitrary, $u \equiv 0$ in Ω and hence on $\overline{\Omega}$ by continuity. This contradicts u being non-trivial and thus the theorem follows.

When L is self-adjoint, we have the following result.

THEOREM 5. *Let* $\partial\Omega \in C^2$ *and* (A) *be positive semidefinite for* (x, ξ) *in* $\Omega \times H^m$. *If there exists a non-trivial* $u \in D$ *such that* (3.2) *and* (3.3) *hold and*

$$f[u] \geq E[u, V], \tag{3.8}$$

then every L*-prepared matrix* $V \in D^m$ *for which* $V^T L V$ *is positive semidefinite and* $V^T A \Delta V$ *is negative definite in* Ω *must be singular at some point in* $\overline{\Omega}$ *unless there exists a constant vector* $r \neq 0$ *such that* $u = Vr$ *on* $\overline{\Omega}$. *If the strict inequality holds in* (3.8), *then* V *must be singular at some point in* $\overline{\Omega}$.

Proof. The proof is similar to that of Theorem 4, except that we use $E[u, V]$, $J[u, V]$ and $J_\omega[u, V]$ instead of $F[u, V]$, $Q[u, V]$ and $Q_\omega[u, V]$ respectively. The second statement is obvious from the proof.

When $m = 1$, Theorem 5 reduces to a result similar to the main result of Dunninger [10] and Theorem 4 deals with the nonself-adjoint case.

We can also consider mixed boundary conditions, for example, of the type

$$u = 0 \quad , \quad a\Delta u + \alpha(x)\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{3.9}$$

where $0 \leqq \alpha \leqq +\infty$ with $\alpha = +\infty$ denoting $\frac{\partial u}{\partial n} = 0$. A typical result that follows immediately from Picone integral identity (2.5) is given below.

THEOREM 6. *Let $\partial\Omega$ be piecewise smooth and* G *satisfy* (3.1). *If there exists a non-trivial $u \in \mathrm{D}$ satisfying* (3.2) *and* (3.9) *such that* (3.4) *holds, then every* L*–prepared matrix* $\mathrm{V} \in \mathrm{D}^m$ *for which* $\mathrm{V}^\mathrm{T}\mathrm{LV}$ *and* $-\mathrm{V}^\mathrm{T}\mathrm{A}\Delta\mathrm{V}$ *are positive semi-definite in Ω must be singular at some point of* $\overline{\Omega}$.

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Added in proof. Related work for certain fourth order partial differential equations and systems of second order partial differential equations was done by G. Cimmino in his papers " Sulle equazioni lineari ellittiche autoaggiunte alle derivate parziali di ordine superiore al secondo, « Memorie della Reale Accademia d'Italia », *1*, 5–15 (1930) " and " Teoremi di confronto fra equazioni o sistemi di equazioni differenziali lineari del second'ordine, « Seminario Matematico della R. Università degli Studi Roma », *14*, 3–24 (1936) " respectively.

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