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Generalizations of Banach's fixed point theorem

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Generalizations of Banach's fixed point theorem.* Nota[*] di TUDOR ZAMFIRESCU, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Questa Nota contiene alcune generalizzazioni del Teorema di Banach sul punto unito, generalizzazioni relative al caso che per ogni m punti si sappia soltanto che vengono contratte almeno k delle loro coppie, m e k essendo numeri fissi, con $k < m(m-1)/2$.

Banach's classical theorem on contractions has been generalized in many ways. We intend to present here another direction in which generalizations are possible; we believe this direction to be new, but it is difficult to guarantee it!

Let M be a metric space and α a real number in $(0, 1)$. We say that a couple of points (x, y) *is contracted* through the function $f: M \to M$ if $x \neq y$ and

$$d(f(x), f(y)) \leq \alpha d(x, y),$$

where d is the distance function on M. We shall also call f an (m, k)-*contraction* if for every m points in M, at least k of the $\binom{m}{2}$ couples determined by them are contracted through f.

Each proposition of the type " each (m, k)-contraction in a complete metric space has a unique fixed point " is a generalization of Banach's theorem if $k < \binom{m}{2}$. However, we easily see that almost always unessential additional assumptions have to be made.

[*] Pervenuta all'Accademia il 29 settembre 1972.

We ask for an (as complete as possible) enumeration of all valid generali-zations of this type (under as weak additional assumptions as possible).

Clearly, not all such propositions are true (at least in the absence of any additional assumptions): Take $n = 3$, $k = 1$, $M = \{0, 1, \alpha, \alpha^2, \alpha^3, \cdots\}$, and $f : M \to M$ defined by

$$f(0) = 0 \quad , \quad f(1) = 1 \quad , \quad f(\alpha^n) = \alpha^{n+1} \qquad (n = 1, \cdots, \infty).$$

We shall present here just a few partial results.

LEMMA 1.   *Each* $(m, (m^2 - 3m + 4)/2)$-*contraction is continuous.*

Proof.   Let $f : M \to M$ be an $(m, (m^2 - 3m + 4)/2)$-contraction, $a \in M$ and $\varepsilon > 0$. If in $S(a, \varepsilon/2m)$ there are less than m points [1], then a is isolated and f is trivially continuous in a. In the other case, choose $m - 1$ (distinct) points $x_1, \cdots, x_{m-1}$ in $S(a, \varepsilon/2m) - \{a\}$.   At least

$$\frac{m^2 - 3m + 4}{2} = \binom{m}{2} - (m - 2)$$

couples from all those determined by $x_1, \cdots, x_{m-1}, a$ are contracted through f, i.e. at most $m - 2$ couples (of distinct points) are not contracted. Now, two cases are possible:

C a s e I. (x_1, a) *is contracted.*

Then, clearly, $f(x_1) \in S(f(a), \varepsilon/2m)$.

C a s e II. (x_1, a) *is not contracted.*

Since each graph on m points having at least $\binom{m}{2} - (m - 2)$ edges is connected, there exists a path (of length less than m) joining x_1 and a in the graph with point-set $x_1, \cdots, x_{m-1}, a$ and edge-set $\{e : e$ is contracted$\}$. Let $x_1, y_1, \cdots, y_p, a$ be this path $(p \leq m - 2)$.   We have

$$d(f(x_1), f(a)) \leq d(f(x_1), f(y_1)) + d(f(y_1), f(y_2))$$
$$+ \cdots + d(f(y_{p-1}), f(y_p)) + d(f(y_p), f(a))$$
$$\leq d(x_1, y_1) + d(y_1, y_2) + \cdots + d(y_{p-1}, y_p) + d(y_p, a)$$
$$\leq 2p \frac{\varepsilon}{2m} + \frac{\varepsilon}{2m} < \frac{p+1}{m} \varepsilon < \varepsilon.$$

Thus, in both cases, $f(x_1) \in S(f(a), \varepsilon)$, which proves the lemma. By $f^n(x)$ we shall mean $f(f^{n-1}(x))$; $f^0(x) = x$.

LEMMA 2.   *Suppose* M *is a metric space and* $f : M \to M$ *is an* $(m, (m^2 - 3m + 4)/2)$-*contraction in* M. *If, for some* $x_0 \in M$, $\{f^n(x_0)\}_{n=0}^{\infty}$

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[1] $S(a, \beta) = \{x \in M : d(x, a) \leq \beta\}$.

has an infinity of distinct points and converges to a point z in M, *then z is a unique fixed point of f.*

Proof. From the continuity of f, assured by Lemma 1, it follows that z is a fixed point for f. Now, suppose z' is another fixed point of f. Then one can find a number p' such that $(z', f^n(x_0))$ is contracted for no $n \geq p'$. Choose $m - 1$ distinct points $f^{p_1}(x_0), f^{p_2}(x_0), \cdots, f^{p_{m-1}}(x_0)$, with $p_i \geq p'$ ($i = 1, \cdots, m - 1$). They plus the point z' admit $m - 1$ non-contracted couples, namely $(z', f^{p_1}(x_0)), \cdots, (z', f^{p_{m-1}}(x_0))$, which contradicts the hypothesis. Hence f has exactly one fixed point.

THEOREM 1. *If the complete metric space* M *has at most finitely many isolated points, f is an $(m, (m^2 - 3m + 4)/2)$-contraction in* M *and, for some $x_0 \in$ M, $\{f^n(x_0)\}_{n=0}^\infty$ has an infinity of distinct points, then f has a unique fixed point.*

Proof. There exists a number r such that $\{f^n(x_0)\}_{n=r}^\infty$ contains no isolated points. Then, suppose $(f^q(x_0), f^{q+1}(x_0))$ is not contracted through f ($q \geq r$). Consider the sequence of distinct points $\{s^n\}_{n=1}^\infty$ converging to $f^{q+1}(x_0)$. By Lemma 1, $f(s^n) \to f^{q+2}(x_0)$. Obviously, one can find again a number p such that $(f^q(x_0), s^n)$ is contracted for no $n \geq p$. Then the m points $f^q(x_0), f^{q+1}(x_0), s^p, s^{p+1}, \cdots, s^{p+m-3}$ admit $m - 1$ non-contracted couples, namely $(f^q(x_0), f^{q+1}(x_0)), (f^q(x_0), s^p), \cdots, (f^q(x_0), s^{p+m-3})$, which contradicts the hypothesis. Hence, A_r denoting the set of all points in $\{f^n(x_0)\}_{n=r}^\infty$, $f | A_r$ is a usual contraction and therefore, by the Picard–Banach contraction principle, $\{f^n(x_0)\}_{n=0}^\infty$ converges to a point $z \in$ M.

Now, from Lemma 2, it follows that z is a unique fixed point of f.

THEOREM 2. *Suppose the metric space* M *has at most finitely many isolated points and f is an $(m, (m^2 - 3m + 4)/2)$-contraction in* M. *If, for some $x_0 \in$ M, $\{f^n(x_0)\}_{n=0}^\infty$ has an infinity of distinct points and also a limit point z in* M, *then z is a unique fixed point of f.*

Proof. Like in the proof of Theorem 1, we find a number r such that $f | A_r$ is a usual contraction, where A_r denotes the set of all points in $\{f^n(x_0)\}_{n=r}^\infty$. Then, like in the standard proof of the Picard–Banach contraction principle, we can show that $\{f^n(x_0)\}_{n=r}^\infty$ is a Cauchy sequence. Hence, this sequence must converge to its limit point z. Now, following Lemma 2, z is a unique fixed point of f.

THEOREM 3. *Suppose* M *is a set in a complete metric space* X *and has at most finitely many isolated points. Let f be an $(m, (m^2 - 3m + 4)/2)$-contraction in* M *and denote $s_x = \{f^n(x)\}_{n=0}^\infty$, for $x \in$ M. If s_{x_0} has an infinity of distinct points for some $x_0 \in$ M, then there exists a point in* X, *to which all s_x with $x \in$ M converge.*

Proof. Like in the proof of Theorem 1, we show that s_{x_0} converges to some point $z \in X$. Take now $y \in M$. Suppose s_y does not converge to z. Two cases are possible:

C a s e   I. *s_y contains finitely many distinct points.* Then s_y has a limit point $w \in M$, different from z. We can find a number $t > 0$ with the property that $f^t(w) = w$. Hence there exists another number $u < t$ such that $d(f^u(w), z) \leq d(f^{u+1}(w), z)$. Obviously, $f^{u+1}(w) \neq z$, otherwise also $f^u(w) = z$ and s_y would converge to z. We choose p', like in the proof of Lemma 2, such that $(f^u(w), f^n(x_0))$ is contracted for no $n \geq p'$. Following further the same argument as in the mentioned proof (of Lemma 2), we find a contradiction.

C a s e   II. *s_y contains infinitely many distinct points.* Then s_y converges to a point in X different from z and there exist two numbers p' and q such that $(f^n(x_0), f^q(y))$ is contracted for no $n \geq p'$. Now, again like in the proof of Lemma 2, we can find a contradiction.

Therefore s_y converges to z and the proof is finished.

Similar are the proofs of the next three theorems; therefore the last two of them we only state.

THEOREM 4. *If M is a complete metric space, f is an $(m, [(m^2 - 2m + 3)/2])$-contraction in M and, for some $x_0 \in M$, $\{f^n(x_0)\}_{n=0}^{\infty}$ has an infinity of distinct points, then f has a unique fixed point.*

Proof. Let

$$B = \{q \geq 0 : (f^q(x_0), f^{q+1}(x_0)) \text{ is not contracted}\}$$

and suppose

$$\text{card } B \geq [m/2].$$

Take a subset C of B with card $C = [m/2]$ and put

$$D = \{q \geq 1 : q - 1 \in C\}.$$

We have obviously card $C = $ card D and therefore

$$\text{card }(C \cup D) \leq 2[m/2] \leq m.$$

Now, if E is a set of positive integers including $C \cup D$ and with card $E = m$, and $F = \{f^q(x_0) : q \in E\}$, then evidently among all couples of distinct points in $F \times F$ there are at least $[m/2]$ which are not contracted. On the other hand, since f is an $(m, [(m^2 - 2m + 3)/2])$-contraction, there are in $F \times F$ at least $[(m^2 - 2m + 3)/2]$ contracted couples, which would mean that $F \times F$ possesses at least

$$[m/2] + [(m^2 - 2m + 3)/2] = \binom{m}{2} + 1$$

couples of distinct points, which is absurd.

Hence card $B < [m/2]$ and we may choose a number r such that $f \mid A_r$ is a usual contraction (A_r being the set of all points in $\{f^n(x_0)\}_{n=r}^{\infty}$). Now, as in the proof of Theorem 1, we conclude that f has a unique fixed point in M.

THEOREM 5. *Suppose* M *is a metric space and* f *is an* $(m, [(m^2 - 2m + 3)/2])$–*contraction in* M. *If, for some* $x_0 \in$ M, $\{f^n(x_0)\}_{n=0}^{\infty}$ *has an infinity of distinct points and also a limit point* z *in* M, *then* z *is a unique fixed point of* f.

THEOREM 6. *Let* M *be a set in a complete metric space* X, f *an* $(m, [(m^2 - 2m + 3)/2])$–*contraction in* M, *and* $s_x = \{f^n(x)\}_{n=0}^{\infty}$, *for* $x \in$ M. *If* s_{x_0} *has an infinity of distinct points for some* $x_0 \in$ M, *then there exists a point in* X, *to which all* s_x *with* $x \in$ M *converge*.