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Cubical polyhedra and homotopy, III

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Topologia algebrica. — *Cubical polyhedra and homotopy, III.* Nota (*) di JÓSEF BLASS e WŁODZIMIERZ HOLSZTYŃSKI, presentata dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota, che fa seguito ad altre due apparse con lo stesso titolo in questi Rendiconti [1], vengono discusse proprietà combinatorie di poliedri cubici, mostrando fra l'altro che ogni poliedro simpliciale risulta omeomorfo ad un poliedro cubico.

§ 1. Throughout this paper I will denote the closed interval $[-1, 1]$, I^n denotes the n-dimensional cube. In particular I^0 is a one-point space.

(1.1) THEOREM. *Every finite simplicial polyhedron is homeomorphic to a cubical polyhedron.*

This theorem can be concluded from the following:

(1.2) LEMMA. *There is a homeomorphism of the n-dimensional simplex Δ^n onto the n-dimensional cube I^n under which the image of the body of any subcomplex of Δ^n is a cubical polyhedron.*

Proof. Set $E_0 = (0, \cdots, 0)$, $E_1 = (1, 0, \cdots, 0), \cdots, E_n = (0, 0, \cdots, 0, 1)$. We define $h' : \Delta^n \to [0, 1]^n$ as follows:

$$(1.3) \qquad \begin{cases} h'(E_0) = E_0 \\ h'(x) = \dfrac{x_1 + \cdots + x_n}{\max(x_1, \cdots, x_n)}\, x . \end{cases}$$

We define $h : \Delta^n \to I^n$ as follows:

$$(1.4) \qquad h(x) = 2 \cdot h'(x) - (1, 1, \cdots, 1) .$$

The map h is a homeomorphism. We will show that h transforms faces of Δ^n onto cubical polyhedra. First, let V be a vertex of Δ^n. If $V = E_0$, then $h(V) = (-1, -1, \cdots, -1)$. If $V = (0, 0, \cdots, 0, \underset{k\text{-th place}}{1}, 0, \cdots, 0)$, then $h(V) = (-1, -1, \cdots, \underset{k\text{-th place}}{-1, 1}, -1, \cdots, -1)$. We have also $h(\Delta^n) = I^n$. Thus the theorem holds for $n = 1$.

Assume now that our theorem is true for Δ^{n-1} $(n > 1)$. Let V be a face of Δ^n. If $V = \Delta^n$ then $h(V) = I^n$. Assume that $V \neq \Delta^n$. We can find

(*) Pervenuta all'Accademia il 30 agosto 1972.

an $(n-1)$–dimensional face W of Δ^n such that $V \subset W$. We have to consider two cases:

(i) $W = \Delta^n_\varkappa = \{x \in \Delta^n ; x_\varkappa = 0\}$ for certain $\varkappa = 1, \cdots, n$. We define $i : \Delta^{n-1} \to \Delta^n$, and $j : I^{n-1} \to I^n$ as follows:

$$i(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_{\varkappa-1}, 0, x_\varkappa, x_{\varkappa+1}, \cdots, x_{n-1})$$

$$j(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_{\varkappa-1}, -1, x_\varkappa, x_{\varkappa+1}, \cdots, x_{n-1}) .$$

It is easy to check that the following diagram commutes.

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "i"] \arrow[d, "h"'] & \Delta^n \arrow[d, "h"] \\ I^{n-1} \arrow[r, "j"'] & I^n \end{tikzcd}$$

Since i is simplicial and j sends faces of I^{n-1} onto the faces of I^n, by induction, $h(V)$ is a cubical polyhedron.

(ii) $W = \{x \in \Delta^n : x_1 + \cdots + x_n = 1\}$. If $V \neq W$ then $V \subset \Delta^n_\varkappa$ for some $\varkappa$ and this case was covered by (i). Assume thus that $V = W$. Then $h'(V) = h'(W) = \{x \in [0, 1]^n : \max(x_i : i = 1, \cdots, n) = 1\}$ and $h(V) = \{x \in I^n : \max(x_i : i = 1, \cdots, n) = 1\}$. Thus $h(V)$ is a cubical subpolyhedron of I^n.

(1.4) THEOREM. *There is a simplicial map which is not topologically equivalent to any map induced by the cubical morphism.*

To show this we need the following obvious lemma.

(1.5) LEMMA. *Assume that* $V \subset I^A$ *is a union of two one-dimensional faces* V_1 *and* V_2 *such that* $\varnothing \neq V_1 \cap V_2 \neq V_1$. *Let* q *be a cubical morphism of* V *into* $W \subset I^{A_1}$. *Then one of the following is true*:

(i) $f_q(V)$ *consists of one point*;

(ii) $f_q(V_i)$ *consists of one point and* f_q *is a homeomorphism on* V_{3-i} *for* $i = 1, 2$;

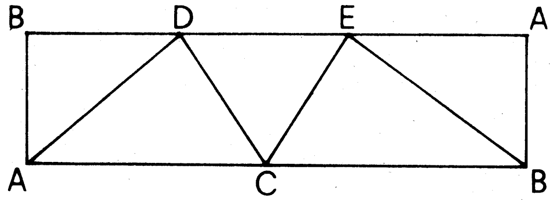
(iii) $f_q : V \to f_q(V)$ *is a homeomorphism.*

Proof of the theorem (1.4). Let f be a simplicial map of $([0,1] \times \{0\}) \cup \cup (\{0\} \times [0,1])$ onto $[0, 1]$ given by

$$f(x, y) = \max(x, y) .$$

If the map $f_q : V \to W$ induced by a cubical morphism q was topologically equivalent to f then f_q would not satisfy Lemma (1.5).

(1.6) *Example.* Mobius strip M admits a simplicial triangulation with 5 vertices A, B, C, D, E.



Thus it is homeomorphic to a cubical polyhedron in I^4.

(1.7) *Example.* Real projective plane admits a simplicial triangulation with 6 vertices (see [2]). Thus it is homeomorphic to a cubical polyhedron in I^5.

§ 2. In this section we discuss cell polyhedra, whose cells are cubes. We study their relation to cubical polyhedra.

(2.1) DEFINITION. A (finite) cell complex is a pair (X, Σ) consisting of a set X and a finite set Σ of functions $\sigma : I^n \to X$ $(n = n(\sigma))$ such that the following conditions are satisfied

(i) every $\sigma \in \Sigma$ is injective;

(ii) for every $\sigma : I^n \to X$ in Σ, every $\varkappa = 1, \cdots, n$ and $\varepsilon = \pm 1$ $(\sigma \circ i_{\varkappa,\varepsilon,n} : I^{n-1} \to X) \in \Sigma$, where $i_{\varkappa,\varepsilon,n} : I^{n-1} \to I^n$ is given by $i_{\varkappa,\varepsilon,n}(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_{\varkappa-1}, \varepsilon, x_{\varkappa+1}, \cdots, x_{n-1})$;

(iii) for every $\mu : I^m \to X$ and $\nu : I^n \to X$ in Σ either $\mu(I^m) \cap \nu(I^n) = \varnothing$ or $\mu^{-1}(\nu(I^n))$ and $\nu^{-1}(\mu(I^m))$ are faces of I^m and I^n respectively and $\nu^{-1} \circ \mu \,|\, \mu^{-1}(\nu(I^n)) : \mu^{-1}(\nu(I^n)) \to \nu^{-1}(\mu(I^m))$ is affine (1);

(iv) if $(\sigma : I^n \to X)$ Σ and $\varphi : I^n \to I^n$ is an affine bijection then $\sigma \circ \varphi \in \Sigma$.

(2.2) DEFINITION. A function $f : X \to X'$ is said to be a cell map of a cell complex (X, Σ) into a cell complex (X', Σ') if for every $(\sigma : I^n \to X) \in \Sigma$ there exists $(\sigma' : I^m \to X') \in \Sigma'$ and a cubical morphism $q : I^n \to I^m$ such that $f \circ \sigma = \sigma' \circ f_q$.

In this way the category CELL of cell complexes is defined. To every cubical polyhedron $X \subset I^A$ (A–finite) a cell complex (X, Σ_X) is canonically assigned. Σ_X is defined as the collection of all affine bijections of cubes I^n onto faces of X. Every cubical morphism becomes a cell map. We have thus obtained a forgetful functor from the category of finite cubical polyhedra QP_0 into CELL.

(1) This affine map is bijective.

We will be considering every cell complex (X, Σ) together with the unique Hausdorff topology on X under which every element of Σ is continuous. Thus the topology of a cubical polyhedron $X \subset I^A$ (A–finite) coincides with the topology of (X, Σ_X).

In general it is much easier to represent a polyhedron as a cell complex than as a cubical polyhedron. We will show below how to embed a large family of cell complexes into cubes, so that they become cubical polyhedra. In general this is impossible. If (X, Σ) is a cell complex such that X is a geometric polyhedron in a Euclidean space, we will assume that all the elements of Σ are affine. Thus to determine (X, Σ) it suffices to know the sets $\sigma(I^n)$ for all $\sigma \in \Sigma$. We will call sets $\sigma(I^n)$ cells.

(2.3) DEFINITION. A cell map $f: X \to X'$ of (X, Σ) into (X', Σ) is said to be cell embedding if f is injective and for every $\sigma \in \Sigma f \circ \sigma \in \Sigma'$.

Obviously the composition of cell embedding is a cell embedding.

(2.4) LEMMA. *Let the cell complex* $I_\varkappa$ *be an interval divided into* $\varkappa - 1$ *subintervals (and* $\varkappa$ *vertices).* $I_\varkappa$ *can be embedded into the cell complex* I^n *iff* $\varkappa \leq 2^n$.

Proof. The condition $\varkappa \leq 2^n$ is necessary since I^n has only 2^n vertices. We will show that it is sufficient. If $n = 1$ this is true. Thus let $n > 1$ and suppose that the assertion holds for $n - 1$. Since $2 \leq \varkappa \leq 2^n$ we can write $\varkappa = \varkappa_1 + \varkappa_2$ for some natural $\varkappa_1, \varkappa_2 \leq 2^{n-1}$. The cell complex $I_\varkappa$ can be represented as a union of subcomplexes $I_{\varkappa_1}, I_{\varkappa_2}$ and of an interval joining one of the ends v of $I_{\varkappa_1}$ with one of the ends w of $I_{\varkappa_2}$. By inductive hypothesis $I_{\varkappa_1}$ can be embedded into $I^n_{n,-1}$ and $I_{\varkappa_2}$ into $I^n_{n,1}$ (2). We can choose embeddings so that v is sent into $(1, 1, \cdots, -1)$, and w into $(1, \cdots, 1, 1)$. Thus there is a common extension to an embedding of $I_\varkappa$ into I^n.

(2.5) DEFINITION. The cell product $(X, \Sigma) \times (X', \Sigma')$ is defined as $(X \times X', \Sigma'')$, where Σ'' consists of all functions of the form $(\sigma \times \sigma') \circ \varphi : I^{n+m} \to X \times X'$, where $\varphi : I^{n+m} \to I^{n+m}$ is an arbitrary affine bijection, $(\sigma : I^n \to X) \in \Sigma$ and $(\sigma' : I^m \to X') \in \Sigma'$.

The above product is the categorical product in the category CELL.

(2.6) LEMMA. *If cell complexes* X, Y *can be cell embedded into* X′, Y′ *respectively, then* $X \times Y$ *can be cell embedded into* $X' \times Y'$.

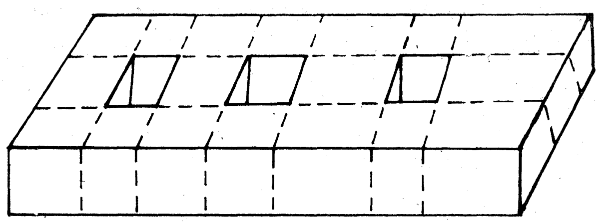
Since the cell product of cubes is a cube by lemma (2.4) we obtain the following

(2.7) THEOREM. *If a cell polyhedron* X *can be cell embedded into* $I_{(\varkappa_1, \cdots, \varkappa_r)} = I_{\varkappa_1} \times \cdots \times I_{\varkappa_r}$ *with* $\varkappa_i \leq 2^{n_i}$ $(i = 1, \cdots, r)$ *then* X *can be cell embedded into* $I^{n_1 + \cdots + n_r}$.

(2) $I^n_{n,\varepsilon} = \{x \in I^n ; x_1 = (\alpha, \cdots, \alpha_{n-1}, \varepsilon)\}$ $\varepsilon = \pm 1$.

(2.8) *Example.* n–sphere S^n is homeomorphic to a cubical polyhedron in I^{n+1}.

(2.9) *Example.* Two-dimensional surface of genus g can be embedded in $I \times I_4 \times I_{2g+2}$ $(I = I_2)$.



Thus it is homeomorphic to a cubical polyhedron in I^n, where

$$n = \begin{cases} 3 + l & \text{if } \; g = 2^l \\ 4 + [\ln_2 g] & \text{if } \; g = 2 \quad \text{for any } l . \end{cases}$$

§ 3. EULER CHARACTERISTIC OF FIBRATIONS. Whenever the product is involved, the cubical approach seems to be more convenient than the simplicial one. As an example we will get an elementary proof of the multiplicative formula for the Euler characteristic of a twisted product.

To every cell complex X a formal polynomial $p(X)$ is assigned

$$p(X) = \sum_{n=0}^{\infty} l_n I^n , \tag{3.1}$$

where l_n is the number of n–dimensional cells of X.

It is easy to show that

$$p(X \times Y) = \sum_{n=0}^{\infty} \sum_{i \geq 0}^{\infty} l_i \cdot l'_{n-i} \, I^n = p(X) \cdot p(Y) . \tag{3.2}$$

More generally, let B, E, F be cell complexes and let $f : E \rightarrow B$ be a map such that for every cell $\sigma : I^n \rightarrow B$, $f^{-1}(\sigma(I^n))$ is a cell subcomplex of E isomorphic to $F \times I^n$, then

$$p(E) = p(B) \cdot p(F) . \tag{3.3}$$

There is a multiplicative homomorphism of the ring of polynomials Z [I] into Z, given by

$$h(p) = \sum_{n} (-1)^n l_n .$$

Then $\chi(X) = h(p(X))$ is the Euler characteristic of X. Thus we have obtained an elementary proof of the following

(3.4) THEOREM. $\chi(E) = \chi(B) \cdot \chi(F)$ *for every twisted cell product* $f : E \underset{F}{\rightarrow} B$.

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[2] W. MASSEY, *Algebraic Topology: An Introduction*, Harcourt, Brace and World, Inc., 1967.