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Special quasi-umbilical hypersurfaces and locus of spheres

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Geometria. — *Special quasi–umbilical hypersurfaces and locus of spheres.* Nota [(*)] di BANG–YEN CHEN e KENTARO YANO, presentata dal Socio B. SEGRE.

RIASSUNTO. — Fra le ipersuperficie di uno spazio euclideo $(n+1)$–dimensionale $(n > 3)$ si introducono quelle che – in base a certe proprietà locali – sono da dirsi quasi–ombelicali, totalmente ombelicali, o quasi–ombelicali speciali, e si dimostra che le ultime risultano luoghi di ∞^1 $(n-1)$–sfere. Si dànno poi condizioni necessarie e sufficienti affinchè un luogo siffatto risulti uno spazio piatto dal punto di vista conforme.

In this paper, we shall introduce and study what we call special quasi-umbilical hypersurfaces. In § 1, we give some formulas and definitions which we use later. In § 2, we shall prove that every special quasi-umbilical hypersurface V_n of a euclidean space E_{n+1} is a locus of $(n-1)$–spheres. In § 3, we shall prove that a locus of $(n-1)$–spheres in E_{n+1} is a conformally flat space if and only if the unit normal vector field of the hypersurface V_n in E_{n+1}, restricted to the $(n-1)$–spheres, is parallel with respect to the normal bundle of the $(n-1)$–sphere in E_{n+1}.

§ 1. PRELIMINARIES

We consider a hypersurface V_n of an $(n+1)$–dimensional euclidean space E_{n+1} and represent it by

$$X = X(\xi^1, \xi^2, \cdots, \xi^n), \tag{1}$$

where X is the position vector from the origin of E_{n+1} to a point of V_n and $\{\xi^h\}$ is a local coordinate system of V_n. Here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$, $n > 3$.

We put

$$X_i = \partial_i X, \qquad \partial_i = \partial/\xi^i, \tag{2}$$

then X_i are n linearly independent vectors tangent to V_n and the components of the fundamental metric tensor of V_n with respect to $\{\xi^h\}$ are given by $g_{ji} = X_j \cdot X_i$, where the dot denotes the inner product in E_{n+1}.

Let C be a unit normal vector field of V_n in E_{n+1}, and let ∇_j denote the operator of covariant differentiation along V_n with respect to the Levi–Civita

(*) Pervenuta all'Accademia il 14 settembre 1972.

connection. Then the equations of Gauss and Weingarten are respectively written as

$$\nabla_j X_i = h_{ji} C \tag{3}$$

and

$$\nabla_j C = -h_j^{\ i} X_i , \tag{4}$$

where h_{ji} are the second fundamental tensor and $h_j^{\ i} = h_{jt} g^{ti}$, g^{ti} being contravariant components of the first fundamental tensor. If there exist, on the hypersurface V_n, two functions α and β and a unit vector field u_i such that

$$h_{ji} = \alpha g_{ji} + \beta u_j u_i , \tag{5}$$

then V_n is said to be *quasi–umbilical* in E_{n+1}. In particular, if $\beta = 0$ identically, then V_n is said to be *totally umbilical* in E_{n+1}. If $d\alpha \neq 0$ everywhere, then V_n is called a *special quasi-umbilical hypersurface* of E_{n+1}.

The equations of Gauss and Codazzi for V_n are given respectively by

$$K_{kji}^{\ \ \ h} = h_k^{\ h} h_{ji} - h_j^{\ h} j_{ki} \tag{6}$$

and

$$\nabla_k h_{ji} - \nabla_j h_{ki} = 0 , \tag{7}$$

where $K_{kji}^{\ \ \ h}$ is the Riemann–Christoffel curvature tensor of V_n. Denoting by $K_{ji} = K_{tji}^{\ \ \ t}$ and $K = g^{ji} K_{ji}$ the Ricci tensor field and the scalar curvature respectively, we define a tensor field L_{ji} of type $(0, 2)$ by

$$L_{ji} = -\frac{K_{ji}}{n-2} + \frac{K g_{ji}}{2(n-1)(n-2)} . \tag{8}$$

The conformal curvature tensor $C_{kji}^{\ \ \ h}$ is then given by

$$C_{kji}^{\ \ \ h} = K_{kji}^{\ \ \ h} + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k^{\ h} g_{ji} - L_j^{\ h} g_{ki} , \tag{9}$$

where δ_k^h are the Kronecker deltas and $L_k^{\ h} = L_{kt} g^{th}$. A Riemannian manifold V_n is called a conformally flat space if $C_{kji}^{\ \ \ h} = 0$ for $n > 3$. It is well-known that if V_n is conformally flat, then we have

$$\nabla_k L_{ji} - \nabla_j L_{ki} = 0 . \tag{10}$$

§ 2. Special quasi-umbilical hypersurfaces

The main purpose of this section is to prove the following

THEOREM 1. *A special quasi-umbilical hypersurface* V_n $(n > 3)$ *of a euclidean space* E_{n+1} *is a locus of* $(n-1)$*–spheres where an* $(n-1)$*–sphere means a hypersphere or a hyperplane of a euclidean n–space.*

Proof. Suppose that V_n is a special quasi-umbilical hypersurface of E_{n+1}, then there exist, on V_n, two functions α and β and a unit vector field u_j such that

$$h_{ij} = \alpha g_{ji} + \beta u_j u_i \tag{11}$$

and

$$d\alpha \neq 0 \quad \text{everywhere}. \tag{12}$$

From (6) and (11), we obtain

$$K_{ji} = [(n-1)\alpha^2 + \alpha\beta]\, g_{ji} + (n-2)\,\alpha\beta u_j u_i \tag{13}$$

and

$$K = n(n-1)\,\alpha^2 + 2\,(n-1)\,\alpha\beta\,. \tag{14}$$

Therefore, we have from (8)

$$L_{ji} = -\frac{\alpha^2}{2} g_{ji} - \alpha\beta u_j u_i\,. \tag{15}$$

From (6), (9) and (15), we can easily see by a straightforward computation that the conformal curvature tensor $C_{kji}{}^h$ vanishes identically. Hence the hypersurface V_n is conformally flat, in particular (10) holds.

From (11) and (15), we have

$$\alpha h_{ji} + L_{ji} = \frac{\alpha^2}{2} g_{ji}\,. \tag{16}$$

By taking covariant derivative of (16) and applying (7) and (10), we obtain

$$\alpha_k h_{ji} - \alpha_j h_{ki} = \alpha\,(\alpha_k g_{ji} - \alpha_j g_{ki})\,, \tag{17}$$

where $\alpha_k = \nabla_k \alpha$.

Substituting (11) into (17), we find

$$\beta\,(\alpha_k u_j - \alpha_j u_k) = 0\,. \tag{18}$$

Put $U = \{p \in V_n\,;\ \beta \neq 0 \text{ at } p\}$. Then by (12), we obtain from (18),

$$u_j = f\alpha_j \tag{19}$$

on U, where f is a function on U. Consequently we find that

$$h_{ji} = \alpha g_{ji} + \gamma \alpha_j \alpha_i \tag{20}$$

on V_n, where γ is a function on V_n.

Substituting (20) into (7), we find

$$\alpha_k g_{ji} - \alpha_j g_{ki} + \gamma_k \alpha_j \alpha_i - \gamma_j \alpha_k \alpha_i + \gamma\alpha_j (\nabla_k \alpha_i) - \gamma\alpha_k (\nabla_j \alpha_i) = 0\,, \tag{21}$$

where $\gamma_k = \nabla_k \gamma$, from which, by transvecting α^k, we obtain

$$(\alpha_t \alpha^t) g_{ji} + (\gamma_t \alpha^t - 1) \alpha_j \alpha_i - (\alpha_t \alpha^t) \gamma_j \alpha_i + \gamma \alpha_j (\alpha^t \nabla_t \alpha) - \gamma (\alpha_t \alpha^t) \nabla_j \alpha_i = 0 . \tag{22}$$

Equation (22) shows that $\gamma \nabla_j \alpha_i$ is of the form

$$\gamma \nabla_j \alpha_i = g_{ji} + q_j \alpha_i + q_i \alpha_j + \frac{\gamma_t \alpha^t - 1}{\alpha_t \alpha^t} \alpha_j \alpha_i , \tag{23}$$

q_j being a 1–form on V_n.

Now, since $\alpha_i = \nabla_i \alpha$ and $\alpha_t \alpha^t \neq 0$, $\alpha =$ constant defines a family of hypersurfaces in V_n. We represent one of them V_{n-1} by

$$\xi^h = \xi^h (\eta^a)$$

and put

$$B_b{}^h = \partial_b \xi^h , \qquad \partial_b = \partial / \partial \eta^b ,$$

$$N^h = \alpha^h / \sqrt{\alpha_t \alpha^t} , \qquad \alpha^h = \alpha_t g^{th} ,$$

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i$$

and

$$\nabla_c B_b{}^h = K_{cb} N^h ,$$

$\nabla_c B_b{}^h$ denoting the van der Waerden–Bortolotti covariant derivative of $B_b{}^h$ along V_{n-1} and K_{cb} the second fundamental tensor of V_{n-1}. Here and in the sequel, the indices $a , b , c , \cdots$ run over the range $\{1 , 2 , \cdots , n-1\}$.
From

$$\alpha_i B_b{}^i = 0 ,$$

we have, by covariant differentiation along V_{n-1},

$$g_{cb} = \gamma \sqrt{\alpha_t \alpha^t} K_{cb} ,$$

by virtue of (23), which shows that γ never vanishes and

$$K_{cb} = \frac{1}{\gamma \sqrt{\alpha_t \alpha^t}} g_{cb} . \tag{24}$$

Thus we have, by (3), (20) and (24),

$$\nabla_c X_b = \nabla_c (B_b{}^i X_i) = K_{cb} N^i X_i + B_c{}^j B_b{}^i (\nabla_j X_i)$$

$$= \alpha g_{cb} C + \frac{1}{\gamma \sqrt{\alpha_t \alpha^t}} g_{cb} D ,$$

where $D = N^i X_i$. This shows that V_{n-1} is totally umbilical in E_{n+1} and hence V_n is a locus of $(n-1)$–spheres. This completes the proof of the theorem.

§ 3. CONFORMALLY FLAT SPACE

In this section, we shall study the problem converse to that of § 2. Let V_n $(n > 3)$ be a locus of $(n-1)$–spheres V_{n-1} in a euclidean $(n+1)$–space E_{n+1} and V_{n-1} be given by

$$\xi^h = \xi^h(\eta^a).$$

Let C denote the unit normal vector field of V_n in E_{n+1} and also denote by C the restriction of C on V_{n-1}. Let N^h denote the unit normal vector field of V_{n-1} in V_n and put

$$B_b{}^h = \partial_b \xi^h, \qquad \partial_b = \partial/\partial\eta^b,$$

$$X_b = B_b{}^i X_i$$

and

$$\nabla_c B_b{}^h = K_{cb} N^h, \qquad \nabla_c C = -K_c{}^a B_a{}^h,$$

where $K_c{}^a = K_{cb} g^{ba}$. Then we have

$$\nabla_c X_b = \nabla_c (B_b{}^i X_i) = K_{cb} N^i X_i + B_c{}^j B_b{}^i (\nabla_j X_i),$$

that is,

$$\nabla_c X_b = H_{cb} C + K_{cb} D, \tag{25}$$

where $D = N^i X_i$ and

$$H_{cb} = B_c{}^j B_b{}^i h_{ji}, \tag{26}$$

from which

$$B_c{}^j h_j{}^i = H_c{}^a B_a{}^i + H_c N^i, \tag{27}$$

$H_c{}^a$ and H_c being given respectively by

$$H_c{}^a = H_{cb} g^{ba} \quad \text{and} \quad H_c = h_{ji} B_c{}^j N^i,$$

$$\nabla_c C = B_c{}^j \nabla_j C = B_c{}^j (-h_j{}^i X_i),$$

that is

$$\nabla_c C = -H_c{}^a X_a - H_c D \tag{28}$$

by virtue of (27), and

$$\nabla_c D = \nabla_c (N^i X_i) = -K_c{}^a B_a{}^i X_i + B_c{}^j N^i (\nabla_j X_i),$$

that is

$$\nabla_c D = -K_c{}^a X_a + H_c C. \tag{29}$$

(25) are equations of Gauss and (28) and (29) are equations of Weingarten for V_{n-1} in E_{n+1}.

Since V_{n-1} is a sphere, we have

$$H_{cb} = \lambda g_{cb} . \tag{30}$$

Thus, from (26) and (30), we have

$$h_{ji} = \lambda g_{ji} + (\mu_j N_i + \mu_i N_j) + \nu N_j N_i , \tag{31}$$

where

$$\mu_j = h_{ji} N^i \tag{32}$$

and ν is a function.

Now V_n is conformally flat if and only if h_{ji} is of the form [1]

$$h_{ji} = \alpha g_{ji} + \beta u_j u_i . \tag{33}$$

From (31) and (33), we see that V_n is conformally flat if and only if μ_j is proportional to N_j. Thus from (32) we see that V_n is conformally flat if and only if

$$H_c = h_{ji} B_c^j N^i = 0 .$$

Thus we have, from (28),

THEOREM 2. *A locus of $(n-1)$-spheres in E_{n+1} is a conformally flat space if and only if the unit normal vector field of the hypersurface V_n in E_{n+1}, restricted to the $(n-1)$-spheres is parallel with respect to the normal bundle of the $(n-1)$-sphere in E_{n+1}.*

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