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Remarks on fixed points, II

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Trasformazioni funzionali. — *Remarks on fixed points, II.* Nota[(*)]
di SIMEON REICH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Questa Nota, che fa seguito ad una precedente, contiene nuovi risultati sulle trasformazioni non espansive negli spazi di Banach. Si segnalano i Teoremi 1.1, 2.6 e 2.7.

INTRODUCTION

This Note is a sequel to [19]. It contains new results on non-expansive mappings in Banach spaces. The main propositions are Theorems 1.1, 2.6 and 2.7.

1. ITERATIONS

Let D be a non-empty subset of a real Banach space E and let G map D into E. We shall denote by R(G) the range of G, by cl(D) the closure of D, and by I the identity mapping (on D). A mapping $T : D \to E$ will be called non-expansive if $\| Tx - Ty \| \leq \| x - y \|$ for all x and y in D. Let N denote the set of all non-negative integers, and let $\{c_n : n \in N\}$ be a sequence of real numbers which satisfy

$$(1.1) \qquad 0 < c_n \leq 1 \qquad \text{for all} \quad n \in N ;$$

$$(1.2) \qquad \sum_{i=0}^{\infty} c_i \quad \text{diverges.}$$

In the sequel we shall denote $\sum_{i=0}^{n} c_i$ by a_n. Let x_0 belong to C, a closed convex subset of E, and let T be a non-expansive self-mapping of C. Define a sequence $\{x_n : n \in N\}$ by

$$(1.3) \qquad x_{n+1} = (1 - c_n)\, x_n + c_n \, Tx_n \, , \qquad n \in N \, .$$

The behavior of $\{x_n\}$ has been studied in [14], [19] and [20]. Here we intend to use a recent idea of Bruck [4] in order to improve [20, Theorem 2.10].

We shall say that C is a non-expansive retract of E if there is a retraction $P : E \to C$ which is non-expansive. If, in addition, $P(x) = v$ implies that $P(v + t(x - v)) = v$ for all $x \in E$ and $t \geq 0$, then C will be called a sunny non-expansive retract. (We prefer this term to those used by Bruck in [2] and [4] because suns already occur in approximation theory).

If $y \in E$ and $r \geq 0$, then the set $\{x \in E : \| x - y \| \leq r\}$ will be denoted by $B(y, r)$ while $S(y, r)$ will stand for $\{x \in E : \| x - y \| = r\}$. Recall that E is said to be uniformly convex in every direction [6] if given $z \in S(0, 1)$ and $\varepsilon > 0$, there exists a positive δ such that $\frac{1}{2} \|x + y\| \leq 1 - \delta$ for all

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x and y in $S(0, 1)$ which satisfy $x - y = tz$ with $\|t\| \geq \varepsilon$. We refer to [5] for information concerning differentiable norms.

THEOREM 1.1. *Let* C *be a non-empty closed convex subset of a real Banach space* E *which is uniformly convex in every direction. Suppose that the norm of* E *is uniformly Gâteaux differentiable while the norm of its dual* E^* *is Fréchet differentiable. Assume further that* C *is the fixed point set of a non-expansive self-mapping of* E. If $T : C \to C$ *is non-expansive and* $\{x_n\}$ *is defined by* (1.3), *then*

(i) $0 \in R(I - T)$ *if and only if* $\{x_n\}$ *is bounded for every* x_0 *in* C *and every sequence* $\{c_n\}$ *which satisfies* (1.1) *and* (1.2);

(ii) $0 \notin cl(R(I - T))$ *if and only if* $\lim_{n\to\infty} \|x_{n+1}\|/a_n > 0$ *for every* x_0 *in* C *and every sequence* $\{c_n\}$ *which satisfies* (1.1) *and* (1.2);

(iii) $0 \in cl(R(I - T))$, *but* $0 \notin R(I - T)$ *if and only if* $\{x_n\}$ *is unbounded and* $x_{n+1}/a_n \to 0$ *for every* x_0 *in* C *and every sequence* $\{c_n\}$ *which satisfies* (1.1) *and* (1.2).

Proof. By [2, Theorem 1] (or by [3, Theorem 2]) C is a non-expansive retract of E. Since the norm of E is uniformly Gâteaux differentiable its duality mapping is uniformly continuous on bounded subsets of E from the strong topology of E to the weak star topology of E^*. Therefore a slight modification of [4, Theorem 2] implies that C is in fact a sunny non-expansive retract of E. Now we can apply Theorems 1.1, 2.3 and 2.8 of [20] and complete the proof.

Theorem 1.1 is applicable in particular to all Banach spaces E such that both E and E^* are uniformly convex. Zizler has shown that every reflexive separable Banach space E has an equivalent uniformly Gâteaux differentiable norm which induces a Fréchet differentiable norm in E^*. With Zizler's norm E is uniformly convex in every direction [21, p. 201]. Every non-empty closed convex subset of a two-dimensional Banach space E is a sunny non-expansive retract of E [10, Theorem 1] and [4, Theorem 5].

2. CHEBYSHEV CENTERS

Let D be a non-empty subset of a real Banach space E. If $Y = \{y_n : n \in N\}$ is a bounded subset of D, we put $r_m(D, Y) = \inf\{r : \{y_n : n \geq m\} \subset B(x, r)$ for some $x \in D\}$ and $R = r(D, Y) = \lim_{m\to\infty} r_m(D, Y)$. We shall call R the asymptotic radius of Y. If D is convex and boundedly weakly compact, then there exists at least one point z in D such that $\limsup_{n\to\infty} \|z - y_n\| = R$. We shall call z an asymptotic center of Y with respect to D (cf. [12] and [7]). If E is uniformly convex in every direction, then the asymptotic center is unique.

If $x \in D$ we denote the set $\{z \in E : z = x + t(y - x)$ for some $t \geq 0$ and $y \in D\}$ by $I_D(x)$. The interior and boundary of D will be denoted by int(D) and bdy(D) respectively. Note that if D is a closed convex subset of a Banach space, then every point in bdy(D) is a bounding point (cf. [19, Section 2]).

PROPOSITION 2.1. *Let* C *be a non-empty convex boundedly weakly compact subset of a Banach space* E *which is uniformly convex in every direction, and let* $T : C \rightarrow E$ *be non-expansive. Suppose that a bounded sequence* $Y = \{y_n : n \in N\} \subset C$ *satisfies* $y_n - Ty_n \rightarrow 0$. *Then each of the following two conditions implies that* T *has a fixed point.*

$$(2.1) \qquad Tx \in I_C(x) \quad \text{for each} \quad x \in C ;$$

$$(2.2) \qquad r(C, Y) = r(E, Y) .$$

Proof. Let $z \in C$ be the asymptotic center of Y with respect to C. If $z \neq Tz$, then $r(C, Y)$ is positive. If (2.1) holds, there exists $0 \leq t < 1$ such that $w = tz + (1 - t) Tz$ belongs to C. But w is another asymptotic center for Y. This contradiction demonstrates that z must be fixed under T. In case (2.2) holds, the point $\frac{1}{2}(z + Tz)$ shows that $r(E, Y) < r(C, Y)$, a contradiction.

Note that if C is a non-expansive retract of E, then (2.2) indeed holds.

COROLLARY 2.2. *Let* C *be a non-empty convex boundedly weakly compact subset of a Banach space which is uniformly convex in every direction. Let a non-expansive* $T : C \rightarrow E$ *have a bounded range. If* T *satisfies* (2.1), *then it has a fixed point.*

Proof. Combine [19, Proposition 2.5] with the preceding proposition.
This corollary may also be deduced from [19, Corollary 2.2].

COROLLARY 2.3. *Let* C, *a closed subset of a Banach space which is uniformly convex in every direction, have a non-empty interior. Let a non-expansive* $T : C \rightarrow E$ *have a bounded range. Assume that for some* $w \in \operatorname{int}(C)$ T *satisfies*

$$(2.3) \qquad Ty - w \neq m (y - w) \quad \text{for all} \quad y \in \operatorname{bdy}(C) \quad \text{and} \quad m > 1 .$$

Then each of the following two conditions implies that T *has a fixed point.*

(i) C *is convex and boundedly weakly compact, and* $r(C, S) = r(E, S)$ *for all bounded sequences* $S \subset C$;

(ii) T *is the restriction of a non-expansive self-mapping of a reflexive* E.

Proof. Combine [9, Corollary 2.3] with Proposition 2.1. (We have not shown that (ii) implies that T must have a fixed point in C).

Remark. When T is a generalized contraction in the sense of Kirk [13] (that is, for each $x \in C$ there is $\alpha(x) < 1$ such that $\| Tx - Ty \| \leq \alpha(x) \| x - y \|$ for all y in C), the uniform convexity assumption can be omitted in Proposition 2.1 (in fact, in this case $\{y_n\}$ converges to the fixed point of T) and in Corollaries 2.2 and 2.3. (In Corollary 2.2 we need no longer assume that T has a bounded range). It follows that Corollaries 3 and 4 in [15] can be improved. (By the way we observe that Corollary 1 there is a direct consequence of [17, Proposition 3.10] while Corollary 2 is included in [16, Corollary 4]).

Recall that a mapping $A : C \to E$ is said to be accretive if for each positive r, $\| x + rAx - y - rAy \| \geq \| x - y \|$ for all x and y in C. If T is non-expansive, then $I - T$ is accretive.

COROLLARY 2.4. *Let* C, *a closed bounded subset of a reflexive Banach space* E *which is uniformly convex in every direction, have a non-empty interior. Suppose that* T, *a Lipschitzian self-mapping of* E, *satisfies* (2.3) *on* C. *If* $I - T$ *is accretive, then* T *has a fixed point.*

Proof. Choose a positive r so that tT may be a strict contraction where $t = r/(r+1)$. $B = [I + r(I - T)]^{-1}$ is single-valued and non-expansive on E. Its restriction to the image of C under $I + r(I - T)$ satisfies (2.3). Corollary 2.3 yields a fixed point for B which is also fixed under T.

This result partially extends [8, Theorem 2] where it is assumed that both E and E^* are uniformly convex. Its proof is inspired by the proof of Theorem 1 in [8].

Let $(C(E), H)$ denote the space of all non-empty compact subsets of a Banach space E, equipped with the Hausdorff metric. Let $S \subset E$ be non-empty. A function $F : S \to C(E)$ is said to be non-expansive if $H(Fx, Fy) \leq \| x - y \|$ for all x and y in S. Combining an extension of Proposition 2.1 to set-valued mappings with [1, Theorem 1] we obtain the following result.

THEOREM 2.5. *Let* C *be a non-empty convex weakly compact subset of a Banach space* E *which is uniformly convex in every direction, and let* $F : C \to C(E)$ *be non-expansive. If* $Fy \subset C$ *for all* y *in* bdy(C), *then* F *has a fixed point.*

THEOREM 2.6. *Let* C, *a convex boundedly weakly compact subset of a Banach space, possess normal structure. Let* $T : C \to C$ *be non-expansive, and let the sequence* $S = \{x_n : n \in N\}$ *be defined by* (1.3). *If* S *is bounded, then* T *has a fixed point.*

Proof. Let $A(C, S)$ denote the set of all the asymptotic centers of S with respect to C. This set is weakly compact, convex and invariant under T. The result follows by [11].

This theorem, which can be extended to more general Toeplitz iterative processes, solves a problem we raised in [20, Section 1]. It shows that " is uniformly convex in every direction " can be replaced by " has normal structure " in Theorem 1.1.

THEOREM 2.7. *Let* T *be a non-expansive self-mapping of a reflexive Banach space* E *which has normal structure. Let a bounded and closed* $C \subset E$ *have a non-empty interior. If* T *satisfies* (2.3) *on* C, *then it has a fixed point.*

Proof. C contains a sequence $Y = \{y_n : n \in N\}$ which satisfies $y_n - Ty_n \to 0$. $A(E, Y)$ is weakly compact, convex and invariant under T. Again an appeal to [11] completes the proof.

This assertion, which has a bearing on Corollary 2.4, partially answers a question we posed in [18].

Remark. A weakly compact convex subset C of a Banach space has the fixed point property for non-expansive mappings if $A(D, S) \neq D$ for all sequences S in closed convex subsets $D \subset C$ which are not singletons.

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