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## On a theorem of J. B. Diaz and F. T. Metcalf

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**Analisi funzionale.** — *On a theorem of J. B. Diaz and F. T. Metcalf.* Nota (*) di S. P. SINGH (**) e M. I. RIGGIO, presentata dal Corrisp. G. FICHERA.

RIASSUNTO. — La Nota è dedicata al teorema del punto unito, del quale viene ora fornita una versione che migliora taluni risultati dati in precedenza da altri Autori.

Diaz and Metcalf have proved a theorem on the convergence of a sequence of iterates. In this Note we want to prove a similar result under less restricted conditions.

Let $T : X \to X$ be a continuous mapping defined on a metric space $(X, d)$. We will need the following preliminaries:

DEFINITION 1. (C. Kuratowskii [7]). *Let $A \subset X$ be a bounded set. The measure of noncompactness of* A, *denoted by* $\alpha(A)$, *is defined to be the infinium of* $\varepsilon > 0$ *such that* A *admits a finite covering consisting of subsets with diameter less than* $\varepsilon$.

*It is easy to see that*:

(a) $0 \leq \alpha(A) \leq \delta(A)$, *where* $\delta(A)$ *is the diameter of the set* $A \subset X$;

(b) $\alpha(A) = 0 \Longleftrightarrow A$ *is precompact*;

(c) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;

(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, *where* A *and* B *are subsets of* X.

DEFINITION 2. *Let* $T : X \to X$ *be a continuous mapping such that*

$$\alpha(TA) \leq k\alpha(A), \tag{1}$$

*for any bounded subset* $A \subset X$.

(a) *If* $k < 1$ *the mapping* T *is called a k–set-contraction* (see G. Darbo [2]);

(b) *If* $k = 1$ *then* T *is said to be a* 1*–set-contraction*;

(c) *In the case where* $\alpha(TA) < \alpha(A)$ *for* $\alpha(A) > 0$, *the mapping* T *is called densifying* (see [5]).

Obviously, if the mapping T is such that

$$d(Tx, Ty) \leq kd(x, y) \tag{2}$$

for all $x, y$ in X, $0 \leq k < 1$, then T satisfies (1).

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It is worth remarking that all contraction mappings and completely continuous mappings are densifying, as well as the sum of these two types of mappings in Banach spaces. Nonexpansive mappings are 1–set-contractions (see [6]).

We will need the following two theorems:

THEOREM A. (see [6]). *Let* $T : C \to C$ *be a densifying mapping defined on a closed bounded convex subset* C *of a Banach space* X. *Then* T *has at least one fixed point.*

THEOREM B. (see [3]). *Let* $T : X \to X$ *be a continuous mapping of a metric space* X *into itself. Suppose*

(i) F(T) *is nonempty, where* F(T) *is the set of fixed points of* T;

(ii) *for each* $y \in X$, *with* $y \notin F(T)$, *and each* $u \in F(T)$, *one has* $d(Ty, u) < d(y, u)$.

Let $x \in X$. Then, either $\{T^n x\}_{n=0}^{\infty}$ contains no convergent subsequence, or $\lim_{n\to\infty} T^n x$ exists and belongs to F(T).

We prove the following theorem.

THEOREM 1. *Let* $T : C \to C$ *be a densifying mapping defined on a closed bounded convex subset* C *of a strictly convex Banach space* X. *Let* T *satisfy the following condition*

$$(3) \qquad \|Tx - Ty\| \le a\|x - y\| + b(\|Tx - x\| + \|Ty - y\|),$$

*for all* $x, y$ *in* C, *where* $a + 2b \le 1$.

*Then for each* $x$ *in* C, *the Picard sequence starting from* $x$ *and generated by the transformation* $T_\lambda$:

$$(4) \qquad T_\lambda x = \lambda Tx + (1 - \lambda)x, \qquad 0 < \lambda < 1,$$

*converges to a fixed point of* T.

*Proof.* It is clear that $T_\lambda$ is defined on C and $T_\lambda C \subset C$, as C is convex. $T_\lambda$ is densifying: Indeed, let A be a bounded non-precompact subset of C. Then $T_\lambda A = \lambda TA + (1 - \lambda) A$, and hence

$$\begin{aligned} \alpha(T_\lambda A) &\le \lambda\alpha(TA) + (1 - \lambda)\alpha(A) \\ &< \lambda\alpha(A) + (1 - \lambda)\alpha(A) \\ &= \alpha(A). \end{aligned}$$

Moreover, F(T) and $F(T_\lambda)$ coincide for every $\lambda$; and by Theorem A, F(T) (and therefore $F(T_\lambda)$) is nonempty.

For $x \in C$, let $A = \bigcup_{n=0}^{\infty} T_\lambda^n x$; we have $T_\lambda A = \bigcup_{n=1}^{\infty} T_\lambda^n x$.

Then A is an invariant set; $A = \{x\} \cup T_\lambda A$.

Denote by $\bar{A}$ the closure of A. $\bar{A}$ is also invariant; indeed, from the continuity of $T_\lambda$, it follows:

$$T_\lambda \bar{A} \subset \overline{T_\lambda A} \subset \bar{A} .$$

Now, we shall prove $\bar{A}$ is compact. It is sufficient to show $\alpha(A) = 0$, since in a complete metric space (and therefore in a Banach space) the precompact sets are also relatively compact. Suppose $\alpha(A) > 0$, $A = T_\lambda A \cup \{x\}$; then

$$\begin{aligned} \alpha(A) &= \max\{\alpha(T_\lambda A), \alpha(x)\} \\ &= \max\{\alpha(T_\lambda A), 0\} = \alpha(T_\lambda A) . \end{aligned}$$

But this contradicts $T_\lambda$ densifying; hence $\alpha(A) = 0$. $\bar{A}$ is compact. Hence the sequence of iterates has a convergent subsequence. Also X strictly convex and condition (3) imply condition (ii) of Theorem B (see [1] for details). Hence by Theorem B, $\{T_\lambda^n x\}$ converges to a fixed point of T.

The following theorem due to Barbuti and Guerra [1] follows as a corollary from Theorem 1.

THEOREM 2. *If* C *is a closed convex subset of a strictly convex Banach space* X *and* $T : C \to C$ *is a continuous transformation which satisfies condition* (3) *and if* T (C) *is contained in a compact subset* K *of* C *then, for every* $x$ *in* C, *the Picard sequence starting from* $x$ *and generated by the transformation* $T_\lambda$ *defined by* (4) *converges to a fixed point of* T.

*Proof*. As in Theorem 1, $F(T) = F(T_\lambda)$; and by Schauder's Theorem [10], $F(T) \neq \varnothing$, then $F(T_\lambda) \neq \varnothing$. Now T(C) is contained in K, a compact subset of C, therefore $\alpha(TC) = 0$; i.e. T is completely continuous and hence trivially densifying.

Then, for every $y \in C - F(T)$ and $u \in F(T)$ we have

$$\| T_\lambda y - u \| < \| y - u \| .$$

This follows from the fact that T satisfies condition (3) on X and X is strictly convex.

The following theorem of J. B. Diaz and F. T. Metcalf [3] can be derived from Theorem 1 as a corollary.

COROLLARY 1. *Let* X *be a strictly convex Banach space and* C *a closed convex set in* X. *Let* $T : C \to C$ *be a nonexpansive mapping defined in* C *such that* T(C) *is a relatively compact set contained in* C. *Let* $T_\lambda = \lambda I + (1 - \lambda) T$, $0 < \lambda < 1$. *Then for each* $x_0$ *in* C, *the sequence* $\{T_\lambda^n x_0\}$ *converges to a fixed point of* T.

*Remark*. In case $\lambda = 1/2$, we have the result of Edelstein [4].

COROLLARY 2. (W. V. Petryshyn [8]). *Let* X *be a strictly convex Banach space*, C *a closed bounded convex subset of* X, *and* $T : C \to C$ *a densifying and nonexpansive mapping. For each* $\lambda$, *with* $0 < \lambda < 1$, *let* $T_\lambda = \lambda T + (1 - \lambda) I$.

*Then for each* $x_0$ *in* C, *the sequence* $\{x_{n+1}\} = \{T_\lambda^n x_0\}$ *determined by the iteration method* $x_{n+1} = \lambda T x_n + (1 - \lambda) x_n$, $n = 0, 1, 2, \cdots; x_0 \in C$, *converges to a fixed point of* T *in* C.

*Proof.* T is nonexpansive and hence condition (3) is satisfied with $a = 1$, $b = 0$. Since $T_\lambda$ is nonexpansive and X strictly convex, it follows that

$$\| T_\lambda y - u \| < \| y - u \| , \qquad u \in F(T) \quad \text{and} \quad y \in C - F(T) .$$

COROLLARY 3. (J. Reinermann [9]). *If* C *is a closed bounded convex subset of a strictly convex Banach space* X *and* $T : C \to C$ *is nonexpansive and completely continuous, then, for each* $\lambda$, $0 < \lambda < 1$, *and* $x_0 \in C$, *the sequence of iterates* $\{x_{n+1}\} = \{T_\lambda^n x_0\}$ *converges to a fixed point of* T.

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