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On a class of Kolmogorov n—width problems

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Analisi matematica. — *On a class of Kolmogorov n–width problems.* Nota (*) di LES ANDREW KARLOVITZ, presentata dal Socio Straniero A. WEINSTEIN.

RIASSUNTO. — Si caratterizzano i sottospazi estremali di una classe di problemi di Kolmogorov n–larghezza. La non unicità della soluzione viene sottolineata e legata alla teoria max–min degli autovalori.

1. INTRODUCTION

Our main concern is with the characterization of extremal subspaces for a class of Kolmogorov n–width problems. In the course of this we emphasize the nonuniqueness of the extremal subspaces for most of the problems, and the possible applications thereof. The nonuniqueness was first announced by us in [1], and seems to be a matter which is generally not well understood. In all of the discussions known to us, either the uniqueness of the "classical" extremal subspace is erroneously claimed or the question is avoided altogether. In particular, Kolmogorov in [2], wherein he introduced the notion of the n–width, overlooked the existence of non-classical extremal subspaces for certain of the problems he considered. The same oversight has also subsequently been made by others. We give an example herein.

Among other things, we state a new approximation-theoretic criterion (Theorem 1, second part) for locating extremal subspaces. By this criterion, an n–dimensional subspace is extremal if it has a basis which satisfies an orthonormality condition and which approximates, term by term, the natural basis of the n–dimensional "classical" extremal subspace within a specified bound.

We also complete a circle of ideas by showing (Paragraph 3) the relation of the results to the maximum–minimum theory of eigenvalues. In particular, the nonuniqueness of the extremal subspace is related to the nonuniqueness of the subspace which yields the eigenvalue in the maximum-minimum theory. The latter nonuniqueness was first discussed by Weinstein [5], [6], wherein he characterized all of the subspaces which yield the eigenvalue.

We do not state our results in their most general form. A more complete version and all of the proofs will be published elsewhere.

2. n–WIDTHS

Let X be a Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. Let e_i, $i \in$ I, be an *orthonormal basis* for X, where the index set I is either the set of positive integers or, if X has finite dimension, I $= \{1, \cdots, \dim X\}$.

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Let λ_i, $i \in I$, be positive increasing reals, i.e., $0 < \lambda_1$ and $\lambda_i < \lambda_{i+1}$, for $i, i+1 \in I$. Let

$$K = \left\{ x : \sum_{i \in I} (\lambda_i \langle x, e_i \rangle)^2 \leq 1 \right\}. \tag{1}$$

The degree to which all points of K can be approximated by a single n–dimensional linear subspace of X is measured by the constant

$$w_n(K) = \inf_{S_n \in \mathfrak{L}_n} \sup_{x \in K} \inf_{y \in S_n} \|x - y\|,$$

where $\mathfrak{L}_n$ is the family of all n–dimensional linear subspaces of X. The constant $w_n(K)$ is called the *n-width of* K. It was first defined (for general Banach spaces X and arbitrary subsets K) by Kolmogorov [2]. A subspace S_n is said to be *extremal for* K if

$$w_n(K) = \sup_{x \in K} \inf_{y \in S_n} \|x - y\|.$$

The first part of the following theorem characterizes the extremal subspaces of K, and the second part states an approximation-theoretic criterion which is sufficient for a subspace to be extremal. This criterion will be seen to be natural and easily applicable (Remark 3 and Example 1, below).

THEOREM 1. *Let* X *be a Hilbert space and* K *be defined by* (1). *Then for each positive integer n, with n < dimension of* X,

$$w_n(K) = 1/\lambda_{n+1}.$$

Moreover, the n–dimensional subspace spanned by $y_1, \cdots, y_n \in K$ *satisfying the orthonormalization*

$$\sum_{i \in I} \lambda_i^2 \langle y_k, e_i \rangle \langle y_j, e_i \rangle = \delta_{k,j}, \qquad k, j = 1, \cdots, n, \tag{2}$$

is extremal for K *if and only if*

$$\langle y_k, e_{n+1} \rangle = 0, \qquad k = 1, \cdots, n, \tag{3}$$

and

$$(\lambda_{n+1} \|x\|)^2 \leq 1 - \sum_{k=1}^{n} \left[\sum_{i \in I} \lambda_i^2 \langle x, e_i \rangle \langle y_k, e_i \rangle \right]^2, \tag{4}$$

for each $x \in K$ *with* $\langle x, y_k \rangle = 0$, $k = 1, \cdots, n$ *and* $\langle x, e_{n+1} \rangle = 0$.

In particular, if $y_1, \cdots, y_n$ *satisfy* (2), (3) *and*

$$\|y_k - e_k/\lambda_k\| \leq \frac{1}{\lambda_k} \min \left\{ \frac{1}{8} \left(\frac{\lambda_1}{\lambda_n} \right)^2, \left(\frac{\lambda_1}{\lambda_{n+1}} \right)^2, \frac{(\lambda_{n+2})^2 - (\lambda_{n+1})^2}{5\, n\, (\lambda_{n+2})^2} \right\}, \tag{5}$$

then they also satisfy (4), *and hence span an extremal subspace.*

Remark 1. The last part of the theorem states that it is sufficient for y_i to be close to e_i/λ_i, $i = 1, \cdots, n$. The bound (5) is not a necessary one.

Neither is it best possible. Better bounds for $\|y_k - e_k/\lambda_k\|$ can be found which, together with (2) and (3), are sufficient to insure that $y_1, \cdots, y_n$ span an extremal subspace. We state the following improved bounds for a specialized case. Suppose $y_1, \cdots, y_n \in K$ satisfy (2) and (3) and $\langle y_k, e_i \rangle \langle y_j, e_i \rangle = 0$ whenever $k \neq j$, $k, j = 1, \cdots, n$, $i = 1, 2, \cdots$, and $\langle y_k, e_{n+i} \rangle = 0$, $i = 1, \cdots, m - 1$. Suppose

$$\|y_k - e_k/\lambda_k\| \leq \varepsilon/\lambda_k, \qquad k = 1, \cdots, n, \tag{6}$$

where ε satisfies

$$\varepsilon^2 (\lambda_n/\lambda_{n+m})^2 + (2\varepsilon)^{2/3} (\lambda_n/\lambda_{n+m}) + 2\varepsilon + \varepsilon^2/c^2 \leq 1 - (\lambda_{n+1}/\lambda_{n+m})^2,$$

and $c = \min \{\langle e_k, y_k \rangle \lambda_k : k = 1, \cdots, n\}$. Then $y_1, \cdots, y_n$ also satisfy (4), and hence span an extremal subspace.

It follows from Theorem 1 that $e_1, \cdots, e_n$ span an extremal subspace for K. For each n, we refer to this as the "classical" choice. It also follows from Theorem 1 that, in almost all cases, non-classical choices can be made. We express this in the following corollary.

COROLLARY 1. *Let* X *and* K *be as a Theorem 1. Let n be a positive integer. If $2n + 1 \leq$ dimension of* X, *then there exists an n–dimensional subspace S_n which is extremal for* K *and which satisfies*

$$S_n \cap \mathit{span}\{e_1, \cdots, e_n\} = \{0\}.$$

If $n + 2 \leq$ dimension of X, *then there exists an n–dimensional subspace S_n which is extremal for* K *and which satisfies*

$$S_n \neq \mathit{span}\{e_1, \cdots, e_n\}.$$

If $n + 1 =$ dimension of X, *then span $\{e_1, \cdots, e_n\}$ is the only n–dimensional subspace which is extremal for* K.

Remark 2. Kolmogorov [2] was concerned mainly with finding the n–width of sets of the form $M = K + L$, where L is a finite dimensional linear subspace and K is a compact ellipsoid, i.e., a compact linear transform of the unit ball. This includes K of the form (1). In particular, he considered $X = L^2[0, 1]$ and

$$M = \left\{ x : x, \frac{dx}{dt} \in L^2[0, 1], \quad x \text{ absolutely continuous}, \quad \left\| \frac{dx}{dt} \right\| \leq 1 \right\}. \tag{7}$$

Based on the geometry of the situation, he erroneously claimed, for example, that all of the extremal subspaces of M, given by (7), are unique. (In Example 1, below, we show that they are nonunique). The claim was based on the idea that if $n > m =$ dimension of L, then an n–dimensional extremal subspace should be spanned by a basis of L and the $n - m$ largest major axes of the ellipsoid K, provided that these are uniquely defined. The following remark is intended to show in geometric terms why this idea is valid only in some special finite dimensional situations. For this purpose, we may take dimension $L = 0$.

Remark 3. For K given by (1) and $y \in K$, one can make the following geometric observation

$$(8) \quad \sup \{\|z\| : z \in K \cap U(y)\} \leq \sup_{x \in K} \inf_{\alpha} \|x - \alpha y\| \leq \sup \{\|z\| : z \in K \cap V(y)\},$$

where $U(y) = \{x : \langle x , y\rangle = 0\}$ and $V(y) = \left\{x : \sum_{i \in I} \lambda_i^2 \langle x , e_i\rangle \langle y , e_i\rangle = 0\right\}$. This observation has two consequences. On the one hand, if the dimension of $X = 2$, i.e., if K is a plane ellipse, then if $y \neq \mu e_1$, all μ, we readily see that $\sup \{\|z\| : z \in K \cap U\} > 1/\lambda_2$. Since $w_1(K) = 1/\lambda_2$, it follows that there is a unique 1-dimensional extremal subspace, namely the classical one spanned by e_1. On the other hand, if the dimension of $X \geq 3$, i.e., if K is a solid ellipsoid, then we readily compute that if $y \in K$ is chosen so that $\langle y , e_2\rangle = 0$ and y is sufficiently close to e_1/λ_1, then $\sup \{\|z\| : z \in K \cap V\} = 1/\lambda_2$. Hence, by (8), y spans an extremal subspace; and, if $y \neq \mu e_1$, all μ, it is a non-classical choice. Thus if the dimension of $X \geq 3$, 1-dimensional extremal subspaces are *not* uniquely determined. A similar argument can be made for $n \geq 2$.

Example 1. Let M be given by (7). Then M can be expressed as $M = L + K$, where $L = \text{span}\{x_1\}$, with $x_1(t) \equiv 1$, and where $K = \left\{x : \sum_{k=1}^{\infty} (\langle x , \sqrt{2} \cos k\pi t\rangle k\pi)^2 \leq 1 , \langle x , x_1\rangle = 0\right\}$. Clearly, if $n \geq 2$ then $w_n(M) = w_{n-1}(K)$ and the subspace spanned by $x_1, \cos \pi t, \cdots, \cos (n-1)\pi t$ is extremal for M. This is the classical choice. Moreover, if S_{n-1} is an extremal subspace for K then $L + S_{n-1}$ is an extremal subspace for M. Thus we can use Theorem 1 to find a non-classical extremal subspace for M. For example, let $u_1(t)$, $u_2(t)$ be piecewise linear, continuous, real-valued functions defined on $[0, 1]$ with knots at $\{0, 1/6, 5/6, 1\}$. They are defined by: $u_1(0) = u_1(1/6) = 1$, $u_1(5/6) = u_1(1) = 0$, $u_2(0) = u_2(1/6) = 0$, and $u_2(5/6) = u_2(1) = 1$. It now follows readily that the subspace spanned by u_1 and u_2 is extremal for M. For $u_1 + u_2 = x_1$ and $(1/\sqrt{6})(u_1 - u_2) = y_1$ is orthogonal to x_1 and it satisfies (2), (3) and (5), with $X = L^2 \cap \{x : \langle x , x_1\rangle = 0\}$, K as given, and $n = 1$. Thus y_1 spans an extremal subspace for K.

In the so-called finite element methods for approximately solving differential equations, non-classical choices of extremal subspaces can be important. It is often the case that computing the approximate solution in an extremal subspace minimizes the error between the approximate and actual solutions. In such cases, the nonuniqueness can be used to select an extremal subspace in some advantageous manner. For example, as indicated by Example 1, one can expect to gain computational advantage by choosing an extremal subspace which is spanned by functions having small supports. The size of the minimal supports in various situations is yet to be investigated.

3. EIGENVALUES

Let A be a completely continuous, positive definite, symmetric linear operator on X. We denote by $\beta_1 \geq \cdots \geq \beta_n \geq \cdots$ and $u_1, \cdots, u_n, \cdots$ the eigenvalues and eigenvectors of A. By the maximum-minimum theory

$$\max_{S_n \in \mathfrak{L}_n} \min_{x \perp S_n} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \beta_{n+1}, \qquad n = 1, 2, \cdots, \tag{9}$$

where $x \perp S_n$ means $\langle x, y\rangle = 0$ for each $y \in S_n$. The following theorem shows the straightforward relation to n-width problems.

THEOREM 2. *Let* X *and* A *be as above. Define an equivalent inner product* $\langle \cdot, \cdot \rangle_A$ *(and norm* $\|\cdot\|_A$*) by* $\langle x, y\rangle_A = \langle Ax, y\rangle$. *Then the n-dimensional subspace* S_n *achieves the maximum in* (9) *if and only if* S_n *is an extremal subspace of*

$$K = \left\{ x : \sum_{i=1}^{\infty} (1/\beta_i^{1/2} \langle x, u_i/\beta_i^{1/2}\rangle_A)^2 \leq 1 \right\}.$$

If $\beta_i > \beta_{i+1}$, $i = 1, 2, \cdots$, it follows from Theorem 2 and Corollary 2 that non-classical choices exist for the subspaces which achieve the maximum in (9). The first characterization of all of the subspaces which achieve the maximum in (9) was given by Weinstein [5], [6]. Our characterizations discussed here are of a different nature from those of Weinstein. Stenger [3] extended the results of Weinstein to a more general class of operators, and in [4] gave a similar development for the minimum-maximum theory. For the work of Weinstein and Stenger we also refer to [7].

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