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On a Kählerian space with recurrent Bochner curvature tensor

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Geometria differenziale. — *On a Kählerian space with recurrent Bochner curvature tensor.* Nota di Baij Nath Prasad, presentata (*) dal Socio E. Bompiani.

Riassunto. — Uno spazio kähleriano è detto uno spazio RB_n se la curvatura tensoriale di Bochner è ricorrente e se il vettore di ricorrenza è nullo. Lo spazio di Kähler è detto uno spazio PB_n. Alcune proprietà dello spazio PB_n sono state studiate in [5] e [6].

In questa Nota sono studiate le proprietà di Einstein dello spazio RB_n e dello spazio ricorrente di Ricci RB_n come pure l'esistenza di campi vettoriali nello spazio RB_n.

1. Introduction

An $n\,(=2m)$ dimensional Kählerian space is a Riemannian space admitting a structure tensor φ_i^h satisfying

$$(1.1)\qquad a)\ \ \varphi_i^h\,\varphi_h^j = -\delta_i^j,$$

$$b)\ \ \varphi_{ij} = -\varphi_{ji}\qquad (\varphi_{ij} = \varphi_i^h g_{hj}),$$

$$c)\ \ \nabla_j\,\varphi_i^h = 0$$

where ∇_j means the operator of covariant differentiation. Let

$$(1.2)\qquad R_{ijk}{}^h = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\}\left\{ {l \atop jk} \right\} - \left\{ {h \atop jl} \right\}\left\{ {l \atop ik} \right\},$$

$R_{jk} = R_{ijk}{}^i$ and $R = R_{jk}\,g^{jk}$ be the Riemannian curvature tensor, the Ricci tensor and scalar curvature respectively.

Recently, S. Tachibana [1] has defined the Bochner curvature tensor (with respect to a real local coordinate system)

$$(1.3)\qquad K_{ijk}{}^h = R_{ijk}{}^h + \frac{1}{n+4}\,(R_{ik}\,\delta_j^h - R_{jk}\,\delta_i^h + g_{ik}\,R_j^h - g_{jk}\,R_i^h +$$

$$+ S_{ik}\,\varphi_j^h - S_{jk}\,\varphi_i^h + \varphi_{ik}\,S_j^h - \varphi_{jk}\,S_i^h + 2\,S_{ij}\,\varphi_k^h + 2\,\varphi_{ij}\,S_k^h) -$$

$$- \frac{R}{(n+2)(n+4)}\,(g_{ik}\,\delta_j^h - g_{jk}\,\delta_i^h + \varphi_{ik}\,\varphi_j^h - \varphi_{jk}\,\varphi_i^h + 2\,\varphi_{ij}\,\varphi_k^h),$$

where $S_{jk} = \varphi_j^h\,R_{hk}$.

The present paper is concerned with a Kähler space whose Bochner curvature tensor $K_{ijk}{}^h$ is recurrent i.e.

$$(1.4)\qquad \nabla_l\,K_{ijk}{}^h = k_l\,K_{ijk}{}^h$$

for a non zero vector k_l. We shall call the Kähler space satisfying (1.4) an RB_n-space.

(*) Nella seduta del 16 giugno 1972.

A Kählerian space is said to be a recurrent space if its curvature tensor $R_{ijk}{}^h$ satisfies

$$\nabla_l R_{ijk}{}^h = k_l^* R_{ijk}{}^h \tag{1.5}$$

for a non zero vector k_l^*. From (1.1) *c*, (1.3), (1.4) and (1.5) we have

THEOREM (1.1). *Every recurrent space is an* RB_n*–space.*

In order to avoid very complicated calculations, we put

$$(1.6)\qquad a)\quad \Pi_{ij} = \frac{1}{n+4}\left(R_{ij} - \frac{R}{2(n+2)}\, g_{ij}\right),$$

$$b)\quad M_{ij} = \varphi_i^h\, \Pi_{hj} = \frac{1}{n+4}\left(S_{ij} - \frac{R}{2(n+2)}\, \varphi_{ij}\right),$$

$$(1.7)\qquad D_{ijk}{}^h = \Pi_{ik}\,\delta_j^h - \Pi_{jk}\,\delta_i^h + g_{ik}\,\Pi_j^h - g_{jk}\,\Pi_i^h + M_{ik}\,\varphi_j^h - M_{jk}\,\varphi_i^h + \varphi_{ik}\,M_j^h - \varphi_{jk}\,M_i^h + 2M_{ij}\,\varphi_k{}^h + 2\varphi_{ij}\,M_k{}^h .$$

Thus (1.3) can be written as

$$K_{ijk}{}^h = R_{ijk}{}^h + D_{ijk}{}^h . \tag{1.8}$$

Moreover Π_{ij}, M_{ij} and $D_{ijk}{}^h$ satisfy the following conditions

$$\Pi = g^{ab}\,\Pi_{ab} = \frac{R}{2(n+2)}, \tag{1.9}$$

$$M_{ij} = -M_{ji}, \tag{1.10}$$

$$D_{ijkh} = D_{khij} \qquad (D_{ijkh} = D_{ijk}{}^l g_{lh}). \tag{1.11}$$

2. EINSTEIN RB_n–SPACE

Let us suppose that RB_n–space defined by (1.4) is an Einstein space, then the Ricci tensor satisfies

$$R_{ij} = \frac{R}{n}\, g_{ij} \quad , \quad \nabla_j R = 0 . \tag{2.1}$$

From which we have

$$\nabla_l R_{ij} = 0 \quad , \quad \nabla_l S_{ij} = 0 \quad \text{and} \quad S_{ij} = \frac{R}{n}\,\varphi_{ij} . \tag{2.2}$$

Substituting (2.1) and (2.2) in (1.4) we get

$$\nabla_l R_{ijk}{}^h = k_l\left[R_{ijk}{}^h + \frac{R}{n(n+2)}\left(\delta_j^h g_{ik} - g_{jk}\,\delta_i^h + \varphi_{ik}\,\varphi_j^h - \varphi_{jk}\,\varphi_i^h + 2\varphi_{ij}\,\varphi_k^h\right)\right]. \tag{2.3}$$

From (2.2) and the Bianchi identity

$$\nabla_l R_{ijk}{}^h + \nabla_i R_{jlk}{}^h + \nabla_j R_{lik}{}^h = 0 , \tag{2.4}$$

we have

$$\nabla_a R_{ijk}{}^a = 0 .$$

Thus contracting with respect to h and l in (2.3) we have

$$(2.5) \quad k^a R_{ijka} + \frac{R}{n(n+2)} (k_j g_{ik} - k_i g_{jk} + k_a \varphi_{ik} \varphi_j^a - k_a \varphi_{jk} \varphi_i^a + 2 k_a \varphi_{ij} \varphi_k^a) = 0 .$$

Furthermore, substituting (2.3) in (2.4) and then transvecting with respect to k^l, we have

$$(2.6) \quad k^a k_a R_{ijk}{}^h + k^a k_a \frac{R}{n(n+2)} (\delta_j^h g_{ik} - g_{jk} \delta_i^h + \varphi_{ik} \varphi_j^h - \varphi_{jk} \varphi_i^h + 2 \varphi_{ij} \varphi_k^h) +$$

$$+ k_i \left\{ k^a R_{jak}{}^h + \frac{R}{n(n+2)} (k^h g_{jk} - k_k \delta_j^h + k^l \varphi_{jk} \varphi_l^h - k^l \varphi_{lk} \varphi_j^h + 2 k^l \varphi_{jl} \varphi_k^h) \right\} +$$

$$+ k_j \left\{ k^a R_{aik}{}^h + \frac{R}{n(n+2)} (k_k \delta_i^h - k^h g_{ik} + k^l \varphi_{lk} \varphi_i^h - k^l \varphi_{ik} \varphi_l^h + 2 k^l \varphi_{li} \varphi_k^h) \right\} = 0 ,$$

which in view of (2.5) gives

$$(2.7) \quad k^a k_a \left\{ R_{ijk}{}^h + \frac{R}{n(n+2)} (\delta_j^h g_{ik} - g_{jk} \delta_i^h + \varphi_{ik} \varphi_j^h - \varphi_{jk} \varphi_i^h + 2 \varphi_{ij} \varphi_k^h) \right\} = 0 .$$

Since k_l is a non null vector, we have the following

THEOREM (2.1). *If an* RB_n*–space is an Einstein one, then the space is of constant holomorphic sectional curvature (Yano* [4] *page 76).*

Yano [4] has proved that

LEMMA (2.1). *A Kähler space of constant holomorphic sectional curvature is an Einstein space:*

Using the above Lemma and the Theorem (2.1) *we have*

THEOREM (2.2). *A necessary and sufficient condition that an* RB_n*–space be an Einstein space is that the space be of constant holomorphic sectional curvature.*

We have the following Lemmas from [2], [3].

LEMMA (2.2). *The curvature tensor* R_{hijk} *satisfies the identity*

$$(2.8) \quad \nabla_l \nabla_m R_{hijk} - \nabla_m \nabla_l R_{hijk} + \nabla_h \nabla_i R_{jklm} - \nabla_i \nabla_h R_{jklm} +$$
$$+ \nabla_j \nabla_k R_{lmhi} - \nabla_k \nabla_j R_{lmhi} = 0 .$$

LEMMA (2.3). *If* $a_{\alpha\beta}$, b_α *are quantities satisfying*

$$(2.9) \quad a_{\alpha\beta} = a_{\beta\alpha} \quad , \quad a_{\alpha\beta} b_\gamma + a_{\beta\gamma} b_\alpha + a_{\gamma\alpha} b_\beta = 0$$

for $\alpha, \beta, \gamma = 1, 2, \cdots, N$ *then either all the* $a_{\alpha\beta}$ *are zero or all the* b_α *are zero.*

With the help of the above Lemmas we shall prove

THEOREM (2.3). *If an* RB_n*–space is an Einstein space, then either the recurrence vector is gradient, or the space is of constant holomorphic sectional curvature.*

Proof. Differentiating (2.3) covariantly and using (1.1) *c*, (2.1), (2.2) and (2.3) we get

$$\nabla_m \nabla_l R_{ijkh} = (\nabla_m k_l + k_l k_m) H_{ijkh} \tag{2.10}$$

where

$$H_{ijkh} = R_{ijkh} + \frac{R}{n(n+2)} (g_{hj} g_{ik} - g_{jk} g_{hi} + \varphi_{ik} \varphi_{jh} - \varphi_{jk} \varphi_{ih} + 2 \varphi_{ij} \varphi_{kh}).$$

From (2.10) and the identity (2.8) we get

$$k_{lm} H_{ijkh} + k_{ij} H_{khlm} + k_{kh} H_{lmij} = 0, \tag{2.11}$$

where

$$k_{lm} = \nabla_l k_m - \nabla_m k_l .$$

The equation (2.11) is of the form (2.9) since $H_{ijkh} = H_{khij}$. Thus from Lemma (2.3) we have the Theorem (2.3).

Now, we assume that an Einstein RB_n–space is recurrent, then we have $\nabla_l R_{ijk}{}^h = \overset{*}{k}_l R_{ijk}{}^h$ for a non zero vector $\overset{*}{k}_l$, from which we have $\nabla_l R_{ij} = \overset{*}{k}_l R_{ij}$. Substituting (2.2) in this equation, we find $R_{ij} = 0$, which is equivalent in an Einstein space to $R = 0$. Conversely, if an Einstein RB_n–space satisfies $R = 0$, then substituting by (2.3) we get $\nabla_l R_{ijk}{}^h = k_l R_{ijk}{}^h$. Thus we have

THEOREM (2.4). *A necessary and sufficient condition for an Einstein* RB_n*–space be recurrent is that the scalar curvature be equal to zero.*

3. RICCI RECURRENT SPACE

THEOREM (3.1). *In an* RB_n*–space if* $\nabla_l M_{ij} = \overset{*}{k}_l M_{ij}$ *for a non zero vector* $\overset{*}{k}_l$ *and a non zero tensor* M_{ij} *then the tensor* $D_{ijk}{}^h$ *is not equal to zero.*

Proof. Suppose on the contrary that $D_{ijk}{}^h = 0$. Since the space is an RB_n–space we have from (1.8)

$$k_l K_{ijk}{}^h = \Delta_l R_{ijk}{}^h .$$

Contracting the above equation with respect to h and i and using the fact that $K_{ajk}{}^a = 0$ (Tachibana [1]) we have

$$\nabla_l R_{jk} = 0 . \tag{3.1}$$

Since the tensor M_{ij} is recurrent, the equations (1.6) *b*, (1.9) and (1.6) *a* give

$$\nabla_l \Pi_{ij} = \overset{*}{k}_l \Pi_{ij} , \nabla_l \Pi = \overset{*}{k}_l \Pi , \nabla_l R = \overset{*}{k}_l R \quad \text{and} \quad \nabla_l R_{ij} = \overset{*}{k}_l R_{ij} . \tag{3.2}$$

From (3.2) and (3.1) we get $\overset{*}{k}_l R_{jk} = 0$. Since $\overset{*}{k}_l \neq 0$ we have $R_{ik} = 0$, $R = 0$ and hence from (1.6) *a* and (1.6) *b* we have $\Pi_{ij} = 0$, $M_{ij} = 0$ which contradict our condition that M_{ij} is a non zero tensor. Hence the tensor $D_{ijk}{}^h \neq 0$.

THEOREM (3.2). *If an RB_n–space of non vanishing Bochner curvature tensor satisfies the condition*

$$\nabla_l M_{ij} = \overset{*}{k}_l M_{ij} \tag{3.3}$$

for a non zero vector $\overset{*}{k}_l$ and a non zero tensor M_{ij}, then a necessary and sufficient condition that k_l be a gradient is that $\overset{*}{k}_l$ be a gradient.

Proof. Let us assume that (1.4) and (3.3) hold, then we have

$$\nabla_l \Pi_{ij} = \overset{*}{k}_l \Pi_{ij} \tag{3.4}$$

and hence from (1.7)

$$\nabla_l D_{ijk}{}^h = \overset{*}{k}_l D_{ijk}{}^h . \tag{3.5}$$

By virtue of (1.4), (1.8) and (3.5) we have

$$\nabla_l R_{ijk}{}^h = k_l K_{ijk}{}^h - \overset{*}{k}_l D_{ijk}{}^h . \tag{3.6}$$

Differentiating (3.6) covariantly and making use of (1.4) and (3.5) we obtain

$$\nabla_m \nabla_l R_{ijkh} = (k_l k_m + \nabla_m k_l) K_{ijkh} - (\overset{*}{k}_l \overset{*}{k}_m + \nabla_m \overset{*}{k}_l) D_{ijkh} .$$

Substituting this equation in the identity (2.8) we get

$$k_{lm} K_{ijkh} + k_{ij} K_{khlm} + k_{kh} K_{lmij} - (\overset{*}{k}_{lm} D_{ijkh} + \overset{*}{k}_{ij} D_{khlm} + \overset{*}{k}_{kh} D_{lmij}) = 0 , \tag{3.7}$$

where

$$k_{lm} = \nabla_l k_m - \nabla_m k_l \quad \text{and} \quad \overset{*}{k}_{lm} = \nabla_l \overset{*}{k}_m - \nabla_m \overset{*}{k}_l .$$

We now assume that k_l is gradient, then $k_{lm} = 0$. Consequently from (3.7) we have

$$\overset{*}{k}_{lm} D_{ijkh} + \overset{*}{k}_{ij} D_{khlm} + \overset{*}{k}_{kh} D_{lmij} = 0 ,$$

which is of the form (2.9) because of (1.11). Since M_{ij} is a non zero tensor by theorem (3.1) D_{ijkh} is a non zero tensor. Therefore it follows from Lemma (2.3) that $\overset{*}{k}_{lm} = 0$. Hence $\overset{*}{k}_l$ is a gradient vector.

Conversely we assume that $\overset{*}{k}_l$ is a gradient vector, then from (3.7) we have

$$k_{lm} K_{ijkh} + k_{ij} K_{khlm} + k_{kh} K_{lmij} = 0 .$$

Since the space is of non vanishing Bochner curvature tensor we find from Lemma (2.3) that k_l is gradient.

Observing the proof of Theorem (3.2) we state

THEOREM (3.3). *Let an* RB_n*–space satisfy the relation* $\nabla_l M_{ij} = \overset{*}{k}_l M_{ij}$ *for a non zero vector* $\overset{*}{k}_l$ *and a non zero tensor* M_{ij} *and* k_l *be a gradient, then* $\overset{*}{k}_l$ *is also gradient.*

Now we shall prove the following Theorem.

THEOREM (3.4). *If an* RB_n*–space which is not flat satisfies* $R_{ij} = 0$ *then the space is recurrent and the recurrence vector is null.*

Proof. If an RB_n–space satisfies $R_{ij} = 0$, then (1.4) can be written as

$$\nabla_l R_{ijk}{}^h = k_l R_{ijk}{}^h. \tag{3.8}$$

Thus the space is recurrent.

From the Bianchi identity and $R_{ij} = 0$, we get $\nabla_a R_{ijk}{}^a = 0$. Thus contracting (3.8) with respect to h and l we get

$$k_a R_{ijk}{}^a = 0. \tag{3.9}$$

On the other hand from the Bianchi identity and (3.8) we have

$$k_l R_{ijk}{}^h + k_i R_{jlk}{}^h + k_j R_{lik}{}^h = 0. \tag{3.10}$$

Transvecting this equation with k^l and using (3.9) we find $k^a k_a R_{ijk}{}^h = 0$. Since the space is not flat, the recurrence vector is null.

4. PARALLEL VECTOR FIELDS IN RB_n–SPACES

Let us assume that in RB_n–space there exist a parallel vector field v^i i.e.

$$\nabla_l v^i = 0 \tag{4.1}$$

Using the Ricci and Bianchi identities we get

$$v^a R_{ija}{}^h = 0 \quad , \quad v^a R_{ja} = 0 \quad , \quad v^a \nabla_a R_{ijk}{}^h = 0 \quad , \quad v^a \nabla_a R_{ij} = 0 \tag{4.2}$$
$$\text{and} \quad v^a \nabla_a R = 0.$$

Let us define $\bar{v}^i = \varphi^i_h v^h$. Differentiating this covariantly and using (4.1) and (1.1)c we get

$$\nabla_l \bar{v}^i = 0. \tag{4.3}$$

Again using the Ricci and Bianchi identities we get

$$\bar{v}^a R_{ija}{}^h = 0 \quad , \quad \bar{v}^a R_{ja} = 0 \quad , \quad \bar{v}^a \nabla_a R_{ijk}{}^h = 0, \tag{4.4}$$
$$\bar{v}^a \nabla_a R_{ij} = 0 \quad \text{and} \quad \bar{v}^a \nabla_a R = 0.$$

Also we have

$$(4.5) \qquad v^a \nabla_a S_{ij} = 0 \quad \text{and} \quad \bar{v}^a \nabla_a S_{ij} = 0 .$$

Transvecting (1.4) by v^l and using (4.2), (4.4) and (4.5) we have $v^a k_a K_{ijk}{}^h = 0$. Thus

THEOREM (4.1) *If an* RB_n*–space admits a parallel vector field* v^i, *then one of the following cases will occur.*

(i) *The space is of vanishing Bochner curvature tensor.*

(ii) v^a *is orthogonal to* k_a.

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