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The effect of conformal change over some entities in Finsler Space

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Geometria. — *The effect of conformal change over some entities in Finsler Space.* Nota di H. D. Pande e Awdhesh Kumar, presentata (*) dal Socio E. Bompiani.

Riassunto. — Due spazi di Finsler si dicono conformi se tali sono le metriche da essi determinate. Si studiano le relazioni fra enti relativi a due di essi.

1. Conformal Finsler Space

Suppose that the two metric functions $F(x^i, \dot{x}^i)$ and $\bar{F}(x^i, \dot{x}^i)$ defined over the n–dimensional Finsler Space F_n satisfy the following conditions:

a) The function $F(x^i, \dot{x}^i)$ is positively homogeneous of degree one in the $\dot{x}^i$'s

$$\text{i.e.} \quad F(x, k\dot{x}^i) = kF(x, \dot{x}^i) \qquad \text{where} \quad k > 0$$

b) The function $F(x, \dot{x})$ is positive if not all $\dot{x}^i$ vanish simultaneously:

$$F(x^i, \dot{x}^i) > 0 \qquad \text{where} \quad \sum_i (\dot{x}^i)^2 \neq 0 .$$

c) The quadratic form

$$F^2_{\dot{x}^i \dot{x}^j}(x, \dot{x})\, \varepsilon^i \varepsilon^j \equiv \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}\, \varepsilon^i \varepsilon^j$$

is assumed to be positive definite for all variables ε^i's for any line element $(x^i, \dot{x}^i)$.

If the above metric functions $F(x, \dot{x})$ and $\bar{F}(x, \dot{x})$ have two metrics $g_{ij}(x, \dot{x})$ and $\bar{g}_{ij}(x, \dot{x})$, they are said to be conformal if

$$\bar{g}_{ij}(x, \dot{x}) = \psi(x, \dot{x})\, g_{ij}(x, \dot{x}) \tag{1.1 a}$$

where $\psi(x, \dot{x})$ is a factor of proportionality.

Equations (1.1 *a*) can also be written in the form [1] (1),

$$\bar{g}_{ij}(x, \dot{x}) = e^{2\sigma} g_{ij}(x, \dot{x}) \tag{1.1 b}$$

where,

$$\sigma = \sigma(x) = \frac{1}{2} \log \psi ,$$

(*) Nella seduta del 16 giugno 1972.

(1) The numbers in square brackets refer to References at the end of the paper.

similarly we can get

$$\bar{g}^{ij}(x, \dot{x}) = e^{-2\sigma} g^{ij}(x, \dot{x}),$$

and metric function $\overline{F}(x, \dot{x})$ defined by

$$\overline{F}(x, \dot{x}) = \psi(x, \dot{x})^{1/2} F(x, \dot{x}), \tag{1.1 c}$$

can also be written by

$$\overline{F}(x, \dot{x}) = e^{\sigma} F(x, \dot{x}) \tag{1.1 d}$$

where $\bar{g}^{ij}$ are called the contravariant components in $\overline{F}_n(x, \dot{x})$ and thus the space $\overline{F}_n$ is called conformal Finsler Space with the entities $\overline{F}$, $\bar{g}^{ij}$ etc.

The covariant derivative of vector $X^i(x, \dot{x})$ with respect to x^k in the sense of Berwald is given by,

$$X^i_{(k)}(x, \dot{x}) = \partial_k X^i - (\dot{\partial}_j X^i) G^j_k + X^j G^i_{jk}\,^{(2)}, \tag{1.1 e}$$

where,

$$G^i_{hj}(x, \dot{x}) \dot{x}^j = G^i_h \quad \text{and,} \tag{1.1 f}$$

$$G^i_{hjk}(x, \dot{x}) \dot{x}^k = 0 \tag{1.1 g}$$

where G^i_{jk} are the Berwald connection coefficients positively homogeneous of degree zero in its directional arguments.

We have the following entities of conformal Finsler Space:

$$\overline{G}^i(x, \dot{x}) = G^i(x, \dot{x}) - \sigma_m B^{im}(x, \dot{x}), \tag{1.1 h}$$

$$\overline{G}^i_j(x, \dot{x}) = G^i_j(x, \dot{x}) - \sigma_m \dot{\partial}_j B^{im}(x, \dot{x}), \tag{1.1 i}$$

$$\overline{G}^i_{jk}(x, \dot{x}) = G^i_{jk}(x, \dot{x}) - \sigma_m \dot{\partial}_j \dot{\partial}_k B^{im}(x, \dot{x}) \tag{1.1 k}$$

where,

$$\sigma_m \overset{\text{def.}}{=} \partial_m \sigma \quad , \quad B^{ij}(x, \dot{x}) \overset{\text{def.}}{=} \frac{1}{2} F^2 g^{ij} - \dot{x}^i \dot{x}^j.$$

The functions $B^{ij}(x, \dot{x})$ are homogeneous of degree two in $\dot{x}^i$. We shall also use the contraction of the tensors H^i_{jk} and H^i_{hjk} as follows:

$$H_i = H^h_{ih} \quad \text{and} \quad H_{ij} = H^h_{ijh} \tag{1.1 l}$$

where, [1]

$$H^i_{jk}(x, \dot{x}) = \partial_k \dot{\partial}_j G^i - \partial_j \dot{\partial}_k G^i + G^i_{kr} \dot{\partial}_j G^r - G^i_{rj} \dot{\partial}_k G^r \tag{1.1 m}$$

(2) $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial \dot{x}^i$.

and

$$\mathrm{H}^i_{hjk}(x\,,\dot{x}) = \dot{\partial}_k\,\mathrm{H}^i_{hj}(x\,,\dot{x})$$

(1.1 n) $$\mathrm{H}^i_{hjk} = \partial_k \mathrm{G}^i_{hj} - \partial_j \mathrm{G}^i_{hk} + \mathrm{G}^r_{hj}\mathrm{G}^i_{rk} - \mathrm{G}^r_{hk}\mathrm{G}^i_{rj} + \mathrm{G}^i_{rhk}\dot{\partial}_j \mathrm{G}^r - \mathrm{G}^i_{rhj}\dot{\partial}_k \mathrm{G}^r .$$

THEOREM 1. *Under the conformal transformation we have:*

(1.1) $$\begin{aligned}
&\overline{\mathrm{H}}^r_{ik(l)} + \overline{\mathrm{H}}^r_{kl(i)} + \overline{\mathrm{H}}^r_{li(k)} = 2\,[\{((\partial_i \mathrm{G}^r_{n[k}) \mathrm{G}^n_{l]} + (\partial_k \mathrm{G}^r_{n[l}) \mathrm{G}^n_{i]} +\\
&+ (\partial_l \mathrm{G}^r_{n[i}) \mathrm{G}^n_{k]}) + ((\partial_k \mathrm{G}^n_{[i}) \mathrm{G}^r_{l]n} + (\partial_l \mathrm{G}^n_{[k}) \mathrm{G}^r_{i]n} + (\partial_i \mathrm{G}^n_{[l}) \mathrm{G}^r_{k]n})\} +\\
&+ \mathrm{G}^m_i (\partial_{[l} \mathrm{G}^r_{k]m} - \mathrm{G}^n_{[l} \mathrm{G}^r_{k]mn} + 2\,\mathrm{G}^n_{m[k} \mathrm{G}^r_{l]n}) + \mathrm{G}^r_{km} (\partial_{[l} \mathrm{G}^m_{i]} - 2\,\mathrm{G}^m_{n[i} \mathrm{G}^n_{l]}) -\\
&- \mathrm{G}^m_k (\partial_{[l} \mathrm{G}^r_{i]m} - \mathrm{G}^n_{[l} \mathrm{G}^r_{i]mn} + 2\,\mathrm{G}^n_{m[i} \mathrm{G}^r_{l]n}) - \mathrm{G}^r_{mi} (\partial_{[l} \mathrm{G}^m_{k]} - 2\,\mathrm{G}^m_{n[k} \mathrm{G}^n_{l]})\} -\\
&- \{(\partial_l \sigma_j - \sigma_n \mathrm{G}^n_{jl}) (\partial_{[k} \mathrm{B}^{rj}_{i]} + \mathrm{G}^r_{m[k} \mathrm{B}^{mj}_{i]} + \mathrm{G}^m_{[i} \mathrm{B}^{rj}_{k]m}) +\\
&+ (\partial_i \sigma_j - \sigma_n \mathrm{G}^n_{ji}) (\partial_{[l} \mathrm{B}^{rj}_{k]} + \mathrm{G}^r_{m[l} \mathrm{B}^{mj}_{k]} + \mathrm{G}^m_{[k} \mathrm{B}^{rj}_{l]m}) +\\
&+ (\partial_k \sigma_j - \sigma_n \mathrm{G}^n_{jk}) (\partial_{[i} \mathrm{B}^{rj}_{l]} + \mathrm{G}^r_{m[i} \mathrm{B}^{mj}_{l]} + \mathrm{G}^m_{[l} \mathrm{B}^{rj}_{i]m})\} -\\
&- \sigma_j \{((\partial_i \mathrm{B}^{rj}_{n[k}) \mathrm{G}^n_{l]} + (\partial_k \mathrm{B}^{rj}_{n[l}) \mathrm{G}^n_{i]} + (\partial_l \mathrm{B}^{rj}_{n[i}) \mathrm{G}^n_{k]}) +\\
&+ ((\partial_k \mathrm{B}^{nj}_{[i}) \mathrm{G}^r_{l]n} + (\partial_l \mathrm{B}^{nj}_{[k}) \mathrm{G}^r_{i]n} + (\partial_i \mathrm{B}^{nj}_{[l}) \mathrm{G}^r_{k]n}) +\\
&+ ((\partial_k \mathrm{B}^{rn}_{[i}) \mathrm{G}^j_{l]n} + (\partial_l \mathrm{B}^{rn}_{[k}) \mathrm{G}^j_{i]n} + (\partial_i \mathrm{B}^{rn}_{[l}) \mathrm{G}^j_{k]n})\} +\\
&+ \mathrm{G}^r_{km} (\partial_{[l} \mathrm{B}^{mj}_{i]} - \mathrm{G}^n_{[l} \mathrm{B}^{mj}_{i]n} + \mathrm{B}^{nj}_{[i} \mathrm{G}^m_{l]n} + \mathrm{B}^{mn}_{[i} \mathrm{G}^j_{l]n}) +\\
&+ \mathrm{B}^{mj}_i (\partial_{[l} \mathrm{G}^r_{k]m} - \mathrm{G}^n_{[l} \mathrm{G}^r_{k]mn} + 2\,\mathrm{G}^n_{m[k} \mathrm{G}^r_{l]n}) + \mathrm{G}^m_i (\partial_{[l} \mathrm{B}^{rj}_{k]m} - \mathrm{G}^n_{[l} \mathrm{B}^{rj}_{k]mn} +\\
&+ \mathrm{G}^r_{n[l} \mathrm{B}^{nj}_{k]m} + \mathrm{B}^{rn}_{m[k} \mathrm{G}^j_{l]n} - \mathrm{B}^{rj}_{n[k} \mathrm{G}^n_{l]m}) + \mathrm{B}^{rj}_{km} (\partial_{[l} \mathrm{G}^m_{i]} - 2\,\mathrm{G}^m_{n[i} \mathrm{G}^n_{l]}) -\\
&- \mathrm{G}^m_k (\partial_{[l} \mathrm{B}^{rj}_{i]m} - \mathrm{G}^n_{[l} \mathrm{B}^{rj}_{i]mn} + \mathrm{B}^{nj}_{m[i} \mathrm{G}^r_{l]n} + \mathrm{B}^{rn}_{m[i} \mathrm{G}^j_{l]n} - \mathrm{B}^{rj}_{n[i} \mathrm{G}^n_{l]m}) -\\
&- \mathrm{B}^{rj}_{im} (\partial_{[l} \mathrm{G}^m_{k]} - 2\,\mathrm{G}^m_{n[k} \mathrm{G}^n_{l]}) - \mathrm{G}^r_{mi} (\partial_{[l} \mathrm{B}^{mj}_{k]} - \mathrm{G}^n_{[l} \mathrm{B}^{mj}_{k]n} +\\
&+ \mathrm{B}^{nj}_{[k} \mathrm{G}^m_{l]n} + \mathrm{B}^{nm}_{[k} \mathrm{G}^j_{l]n}) - \mathrm{B}^{mj}_k (\partial_{[l} \mathrm{G}^r_{i]m} - \mathrm{G}^n_{[l} \mathrm{G}^r_{i]mn} + 2\,\mathrm{G}^n_{m[i} \mathrm{G}^r_{l]n})\} +\\
&+ \sigma_j \sigma_s \{\mathrm{B}^{rj}_{km} (\mathrm{B}^{ns}_{[i} \mathrm{G}^m_{l]n} + \partial_{[l} \mathrm{B}^{ms}_{i]} - \mathrm{B}^{ms}_{n[i} \mathrm{G}^n_{l]} + \mathrm{B}^{nm}_{[i} \mathrm{G}^s_{l]n}) +\\
&+ \mathrm{B}^{ms}_i (\partial_{[l} \mathrm{B}^{rj}_{k]m} - \mathrm{G}^n_{[l} \mathrm{B}^{rj}_{k]mn} + \mathrm{B}^{nj}_{m[k} \mathrm{G}^r_{l]n} + \mathrm{B}^{rn}_{m[k} \mathrm{G}^j_{l]n} - \mathrm{B}^{rj}_{n[k} \mathrm{G}^n_{l]m}) -\\
&- \mathrm{B}^{rj}_{mi} (\partial_{[l} \mathrm{B}^{ms}_{k]} - \mathrm{G}^n_{[l} \mathrm{B}^{ms}_{k]n} + \mathrm{B}^{ns}_{[k} \mathrm{G}^m_{l]n} + \mathrm{B}^{nm}_{[k} \mathrm{G}^s_{l]n}) -\\
&- \mathrm{B}^{ms}_k (\partial_{[l} \mathrm{B}^{rj}_{i]m} - \mathrm{G}^n_{[l} \mathrm{B}^{rj}_{i]mn} + \mathrm{B}^{nj}_{m[i} \mathrm{G}^r_{l]n} + \mathrm{B}^{rn}_{m[i} \mathrm{G}^j_{l]n} - \mathrm{B}^{rj}_{n[i} \mathrm{G}^n_{l]m})\} +\\
&+ (\mathrm{B}^{rj}_{m[k} \mathrm{B}^{ms}_{i]}) \{\sigma_j (\partial_l \sigma_s - \sigma_n \mathrm{G}^n_{sl}) + \sigma_s (\partial_l \sigma_j - \sigma_n \mathrm{G}^n_{jl})\} +\\
&+ (\mathrm{B}^{rj}_{m[l} \mathrm{B}^{ms}_{k]}) \{\sigma_j (\partial_i \sigma_s - \sigma_n \mathrm{G}^n_{si}) + \sigma_s (\partial_i \sigma_j - \sigma_n \mathrm{G}^n_{ji})\} +\\
&+ (\mathrm{B}^{rj}_{m[i} \mathrm{B}^{ms}_{l]}) \{\sigma_j (\partial_k \sigma_s - \sigma_n \mathrm{G}^n_{sk}) + \sigma_s (\partial_k \sigma_j - \sigma_n \mathrm{G}^n_{jk})\}] ,
\end{aligned}$$

and

$$
\begin{aligned}
(1.2)\quad & \bar{H}^r_{k(l)} - \bar{H}^r_{l(k)} + \bar{H}^r_{kl(i)}\dot{x}^i = 2\,[\{((\partial_{[l} G^r_{\langle n\rangle}) G^n_{k]} + 2\,(\partial_{[k} G^n)\, G^r_{l]n}) - \\
& - 2\,(\partial_{[k} B^{rm})(\partial_{l]}\sigma_m - \sigma_n G^n_{l]m}) + 2\,G^j(\partial_{[l} G^r_{k]j} - G^n_{[l} G^r_{k]jn} + \\
& + 2\,G^n_{j[k} G^r_{l]n}) + 2\,G^r_{j[k}(\partial_{l]} G^j - G^n_{l]} G^j_n + G^j_{l]n} G^n) - \\
& - 2\,(B^{jm} G^r_{j[k} + G^j B^{rm}_{j[k})(\partial_{l]}\sigma_m - \sigma_n G^n_{l]m})\} + \\
& + 2\,(B^{rm}_{j[k} B^{js})\{\sigma_s(\partial_{l]}\sigma_m - \sigma_n G^n_{l]m}) + \sigma_m(\partial_{l]}\sigma_s - \sigma_n G^n_{l]s})\} - \\
& - \{G^j_{[k}(\partial_{l]} G^r_j + 2\,G^r_{l]n} G^n_j - 2\,G^n_{l]j} G^r_n) + G^r_j(\partial_{[l} G^j_{k]} - 2\,G^j_{n[k} G^n_{l]}) - \\
& - (\partial_{[l}\sigma_m - \sigma_n G^n_{m[l})(B^{rm}_{k]} G^r_j + G^j_{k]} B^{rm}_j)\} + \\
& + (B^{rm}_j B^{js}_{[k})\{\sigma_m(\partial_{l]}\sigma_s - \sigma_n G^n_{l]s}) + \sigma_s(\partial_{l]}\sigma_m - \sigma_n G^n_{l]m})\} + \\
& + \{(\partial_{[l} G^n_{k]}) G^r_n + (\sigma_n G^n_j)(\partial_{[l} B^{rm}_{k]} + G^r_{j[l} B^{jm}_{k]} + G^j_{[k} B^{rm}_{l]j})\} - \\
& - \dot{x}^h\{((\partial_h G^r_{n[l}) G^n_{k]} + (\partial_h G^n_{[k}) G^r_{l]n}) - (\partial_h B^{rm}_{[k})(\partial_{l]}\sigma_m - \sigma_n G^n_{l]m}) + \\
& + ((\partial_{[l} G^r_{k]n}) G^n_h - G^j_{[k}(\partial_{\langle h\rangle} G^r_{l]j}) + G^j_{[k} G^r_{l]jn} G^n_h - G^r_{j[l}(\partial_{\langle h\rangle} G^j_{k]}) + \\
& + G^r_{j[l} G^j_{k]n} G^n_h) + (\partial_h\sigma_j)(\partial_{[l} B^{rm}_{k]} + G^r_{j[l} B^{jm}_{k]} + G^j_{[k} B^{rm}_{l]j}) + \\
& + \sigma_m((\partial_{[k} B^{rm}_{l]n}) G^n_h + G^r_{j[l}(\partial_{\langle h\rangle} B^{jm}_{k]}) - G^r_{j[l} B^{jm}_{k]n} G^n_h + B^{jm}_{[k}(\partial_{\langle h\rangle} G^r_{l]j}) - \\
& - B^{jm}_{[k}(G^r_{l]jn} G^n_h + G^j_{[k}(\partial_{\langle h\rangle} B^{rm}_{l]j}) - G^j_{[k} B^{rm}_{l]jn} G^n_h + B^{rm}_{j[l}(\partial_{\langle h\rangle} G^j_{k]}) - \\
& - B^{rm}_{j[l} G^j_{k]n} G^n_h - (\partial_h B^{rm}_{n[k}) G^n_{l]} + (\partial_h B^{nm}_{[k}) G^r_{l]n} + (\partial_h B^{rn}_{[k}) G^m_{l]n}) - \\
& - \sigma_m\sigma_s(B^{rm}_{j[l}(\partial_{\langle h\rangle} B^{js}_{k]}) - B^{rm}_{j[l} B^{js}_{k]n} G^n_h + B^{js}_{[k}(\partial_{\langle h\rangle} B^{rm}_{l]j}) - \\
& - B^{js}_{[k} B^{rm}_{l]jn} G^n_h) - (B^{rm}_{j[l} B^{js}_{k]})(\sigma_m(\partial_h\sigma_s) + \sigma_s(\partial_h\sigma_j))\} - \\
& - \sigma_m\{(\partial_{[l} B^{rm}_{\langle n\rangle}) G^n_{k]} + 2\,(\partial_{[k} B^{nm}) G^r_{l]n} + 2\,(\partial_{[k} B^{rn}) G^m_{l]n} + \\
& + (\partial_{[l} B^{nm}_{k]}) G^r_n + (\partial_{[l} B^{rn}_{k]}) G^m_n) + G^r_{j[k}(2\,\partial_{l]} B^{jm} - G^n_{l]} B^{jm}_n + \\
& + 2\,B^{nm} G^j_{l]n} + 2\,B^{jn} G^m_{l]n} - B^{nm}_{l]} G^j_n - B^{jn}_{l]} G^m_n) + \\
& + 2\,B^{jm}(\partial_{[l} G^r_{k]j} - G^n_{[l} G^r_{k]jn}) + 2\,G^n_{j[k} G^r_{l]n}) + \\
& + 2\,G^j(\partial_{[l} B^{rm}_{k]j} - G^n_{[l} B^{rm}_{k]jn} + B^{nm}_{j[k} G^r_{l]n} + B^{rn}_{j[k} G^m_{l]n} - B^{rm}_{n[k} G^n_{l]j}) + \\
& + 2\,B^{rm}_{j[k}(\partial_{l]} G^j - G^n_{l]} G^j_n + G^j_{l]n} G^n) - G^r_j(\partial_{[l} B^{jm}_{k]} - B^{rm}_{n[k} G^n_{l]} + \\
& + G^m_{n[l} B^{jn}_{k]} + B^{nm}_{[k} G^j_{l]n}) - B^{jm}_{[k}(\partial_{l]} G^r_j + 2\,G^r_{l]n} G^n_j - 2\,G^n_{l]j} G^r_n) + \\
& + G^j_{[k}(B^{nm}_{l]j} G^r_n + B^{rn}_{l]j} G^m_n - G^r_{l]n} B^{nm}_j - B^{rm}_{l]n} G^n_j - \partial_{l]} B^{rm}_j - \\
& - G^m_{l]n} B^{rn}_j + G^n_{l]j} B^{rm}_n) + B^{rm}_j(\partial_{[l} G^j_{k]} - 2\,G^j_{n[k} G^n_{l]})\} -
\end{aligned}
$$

$$- \sigma_m \sigma_s \{ B_j^{rm} (\partial_{[l} B_{k]}^{js} - B_{n[k}^{js} G_{l]}^n + B_{[k}^{ns} G_{l]n}^j + B_{[k}^{jn} G_{l]n}^s) +$$

$$+ B_{[k}^{js} (\partial_{l]} B_j^{rm} + G_{l]n}^r B_j^{nm} + G_{l]n}^m B_j^{rn} - G_{l]j}^n B_n^{rm} -$$

$$- B_{l]j}^{nm} G_n^r - B_{l]j}^{rn} G_n^m + B_{l]n}^{rm} G_j^n) -$$

$$- B_{j[k}^{rm} (2 \partial_{l]} B^{js} - G_{l]}^n B_n^{js} + 2 G_{l]n}^j B^{ns} + 2 G_{l]n}^s B^{jn} - B_{l]}^{ns} G_n^j + B_{l]}^{jn} G_n^s) -$$

$$- 2 B^{js} (\partial_{[l} B_{k]j}^{rm} - G_{[l}^n B_{k]jn}^{rm} + G_{n[l}^r B_{k]j}^{nm} + B_{j[k}^{rn} G_{l]n}^m - B_{n[k}^{rm} G_{l]j}^n)\} -$$

$$- \{ (B_{j[l}^{rm} B_{k]}^{js}) (\sigma_m \sigma_n G_s^n + \sigma_s \sigma_n G_j^n) \}] .$$

Proof. As we know that H_k^j is a tensor and is also known as "deviation tensor" being defined by [1]

$$H_k^j (x , \dot{x}) = K_{ihk}^j (x , \dot{x}) \dot{x}^i \dot{x}^h = R_{ihk}^j (x , \dot{x}) \dot{x}^i \dot{x}^h . \tag{1.3}$$

With the help of equations (1.3),

$$\dot{x}^i K_{ihk}^j = \partial_k \dot{\partial}_h G^i - \dot{\partial}_k \partial_h G^i - G_{hl}^j \dot{\partial}_k G^l + G_{kl}^j \dot{\partial}_h G^l \tag{1.4}$$

and

$$R_{jhk}^i = K_{jhk}^i + C_{jm}^i K_{rhk}^m \dot{x}^r \tag{1.5}$$

we have

$$H_k^i (x , \dot{x}) = 2 \partial_k G^i - \partial_h \dot{\partial}_k G^i \dot{x}^h + 2 G_{kl}^i G^l - \dot{\partial}_l G^i \dot{\partial}_k G^l \tag{1.6}$$

where $G^i (x , \dot{x})$ are positively homogeneous of degree 2 in $\dot{x}^i$.

(1.7) We know, $H_{jk}^i \overset{\text{def.}}{=} \frac{1}{3} (\dot{\partial}_j H_k^i - \dot{\partial}_k H_j^i)$

(1.8) so that $H_{jk}^i (x , \dot{x}) = \partial_k \dot{\partial}_j G^i - \partial_j \dot{\partial}_k G^i + G_{kr}^i \dot{\partial}_j G^r - G_{rj}^i \dot{\partial}_k G^r .$

Under the conformal transformation defined in equations (1.1 *h*), (1.1 *i*) and (1.1 *k*) we have,

$$\begin{aligned} \bar{H}_k^r (x , \dot{x}) = {} & [2 (\partial_k G^r - (\partial_k B^{rm}) \sigma_m) - (\partial_h G_k^r - (\partial_h B_k^{rm}) \sigma_m) \dot{x}^h + \\ & + 2 \{ G_{kj}^r G^j - \sigma_m (G_{kj}^r B^{jm} + G^j B_{kj}^{rm}) + (B_{kj}^{rm} B^{js} \sigma_m \sigma_s) \} - \\ & - \{ G_j^r G_k^j - \sigma_m (G_j^r B_k^{jm} + G_k^j B_j^{rm}) + (B_j^{rm} B_k^{js} \sigma_m \sigma_s) \}] , \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} \bar{H}_{ik}^r (x , \dot{x}) = {} & [(\partial_k G_i^r - \partial_i G_k^r + G_{km}^r G_i^m - G_{mi}^r G_k^m) - \\ & - \sigma_j (\partial_k B_i^{rj} - \partial_i B_k^{rj} + G_{km}^r B_i^{mj} + G_i^m B_{km}^{rj} - G_{mi}^r B_k^{mj} - G_k^m B_{mi}^{rj}) + \\ & + \sigma_j \sigma_s (B_{km}^{rj} B_i^{ms} - B_{mi}^{rj} B_k^{ms})] . \end{aligned} \tag{1.10}$$

Differentiating covariantly with respect to x^l equations (1.9) and (1.10) in the sense of Berwald we get respectively equations (1.11) and (1.12).

$$
\begin{aligned}
(1.11)\quad \bar{H}^r_{k(l)} = {}& [2\{(\partial_k\partial_l G^r - (\partial_k G^r_n) G^n_l + (\partial_k G^n) G^r_{nl}) - (\partial_k B^{rm})(\partial_l\sigma_m - \sigma_n G^n_{ml}) - \\
& - \sigma_m(\partial_k\partial_l B^{rm} - (\partial_k B^{rm}_n) G^n_l + (\partial_k B^{nm}) G^r_{nl} + (\partial_k B^{rn}) G^m_{nl})\} - \\
& - \dot{x}^h\{(\partial_h\partial_l G^r_k - (\partial_h G^r_{kn}) G^n_l + (\partial_h G^n_k) G^r_{nl} - (\partial_h G^r_n) G^n_{kl}) - \\
& - (\partial_h B^{rm}_k)(\partial_l\sigma_m - \sigma_n G^n_{ml}) - \sigma_m(\partial_h\partial_l B^{rm}_k - (\partial_h B^{rm}_{kn}) G^n_l + \\
& + (\partial_h B^{nm}_k) G^r_{nl} + (\partial_h B^{rn}_k) G^m_{nl} - (\partial_h B^{rm}_n) G^n_{kl})\} + \\
& + 2[G^j(\partial_l G^r_{kj} - G^r_{kjn} G^n_l + G^n_{kj} G^r_{nl} - G^r_{nj} G^n_{kl} - G^r_{kn} G^n_{jl}) + \\
& + G^r_{kj}(\partial_l G^j - G^j_n G^n_l + G^n G^j_{nl}) - (G^r_{kj} B^{jm} + G^j B^{rm}_{kj})(\partial_l\sigma_m - \sigma_n G^n_{ml}) - \\
& - \sigma_m\{G^r_{kj}(\partial_l B^{jm} - B^{jm}_n G^n_l + B^{nm} G^j_{nl} + B^{jn} G^m_{nl}) + \\
& + B^{jm}(\partial_l G^r_{kj} - G^r_{kjn} G^n_l + G^n_{kj} G^r_{nl} - G^r_{nj} G^n_{kl} - G^r_{kn} G^n_{jl}) + \\
& + G^j(\partial_l B^{rm}_{kj} - B^{rm}_{kjn} G^n_l + B^{nm}_{kj} G^r_{nl} + B^{rn}_{kj} G^m_{nl} - B^{rm}_{nj} G^n_{kl} - B^{rm}_{kn} G^n_{jl}) + \\
& + B^{rm}_{kj}(\partial_l G^j - G^j_n G^n_l + G^n G^j_{nl})\} + \\
& + \sigma_m\sigma_s\{B^{rm}_{kj}(\partial_l B^{js} - B^{js}_n G^n_l + B^{ns} G^j_{nl} + B^{jn} G^s_{nl}) + \\
& + B^{js}(\partial_l B^{rm}_{kj} - B^{rm}_{kjn} G^n_l + B^{nm}_{kj} G^r_{nl} + B^{rn}_{kj} G^m_{nl} - B^{rm}_{nj} G^n_{kl} - B^{rm}_{kn} G^n_{jl}) + \\
& + (B^{rm}_{kj} B^{js})\{\sigma_s(\partial_l\sigma_m - \sigma_n G^n_{ml}) + \sigma_m(\partial_l\sigma_s - \sigma_n G^n_{sl})\}] - \\
& - [G^j_k(\partial_l G^r_j - G^r_{jn} G^n_l + G^n_j G^r_{nl} - G^r_n G^n_{jl}) + \\
& + G^r_j(\partial_l G^j_k - G^j_{kn} G^n_l + G^n_k G^j_{nl} - G^j_n G^n_{kl}) - \\
& - (\partial_l\sigma_m - \sigma_n G^n_{ml})(G^r_j B^{jm}_k + G^j_k B^{rm}_j) - \\
& - \sigma_m\{G^r_j(\partial_l B^{jm}_k - B^{jm}_{kn} G^n_l + B^{nm}_k G^j_{nl} + B^{jn}_k G^m_{nl} - B^{jm}_n G^n_{kl}) + \\
& + B^{jm}_k(\partial_l G^r_j - G^r_{jn} G^n_l + G^n_j G^r_{nl} - G^r_n G^n_{jl}) + \\
& + G^j_k(\partial_l B^{rm}_j - B^{rm}_{jn} G^n_l + B^{nm}_j G^r_{nl} + B^{rn}_j G^m_{nl} - B^{rm}_n G^n_{jl}) + \\
& + B^{rm}_j(\partial_l G^j_k - G^j_{kn} G^n_l + G^n_k G^j_{nl} - G^j_n G^n_{kl})\} + \\
& + \sigma_m\sigma_s\{B^{rm}_j(\partial_l B^{js}_k - B^{js}_{kn} G^n_l + B^{ns}_k G^j_{nl} + B^{jn}_k G^s_{nl} - B^{js}_n G^n_{kl}) + \\
& + B^{js}_k(\partial_l B^{rm}_j - B^{rm}_{jn} G^n_l + B^{nm}_j G^r_{nl} + B^{rn}_j G^m_{nl} - B^{rm}_n G^n_{jl}) + \\
& + (B^{rm}_j B^{js}_k)\{\sigma_m(\partial_l\sigma_s - \sigma_m G^n_{sl}) + \sigma_s(\partial_l\sigma_m - \sigma_n G^n_{ml})\}]]
\end{aligned}
$$

and

$$
\begin{aligned}
(1.12)\quad \bar{H}^r_{ik(l)} = {} & [\{(\partial_l \partial_k G^r_i - (\partial_k G^r_{in}) G^n_l + (\partial_k G^n_i) G^r_{nl} - (\partial_k G^r_n) G^n_{il}) - \\
& - (\partial_i \partial_l G^r_k - (\partial_i G^r_{kn}) G^n_l + (\partial_i G^n_k) G^r_{nl} - (\partial_i G^r_n) G^n_{kl}) + \\
& + G^m_i (\partial_l G^r_{km} - G^r_{kmn} G^n_l + G^n_{km} G^r_{nl} - G^r_{nm} G^n_{kl} - G^r_{kn} G^n_{ml}) - \\
& - G^m_k (\partial_l G^r_{im} - G^r_{imn} G^n_l + G^n_{mi} G^r_{nl} - G^r_{mn} G^n_{il} - G^r_{ni} G^n_{ml}) + \\
& + G^r_{km} (\partial_l G^m_i - G^m_{in} G^n_l + G^n_i G^m_{nl} - G^m_n G^n_{il}) - \\
& - G^r_{mi} (\partial_l G^m_k - G^m_{kn} G^n_l + G^n_k G^m_{nl} - G^m_n G^n_{kl}) - (\partial_l \sigma_j - \sigma_n G^n_{jl}) \cdot \\
& \cdot (\partial_k B^{rj}_i - \partial_i B^{rj}_k + G^r_{km} B^{mj}_i + G^m_i B^{rj}_{km} - G^r_{mi} B^{mj}_k - G^m_k B^{rj}_{mi}) - \\
& - \sigma_j \{(\partial_k \partial_l B^{rj}_i - (\partial_k B^{rj}_{in}) G^n_l + (\partial_k B^{nj}_i) G^r_{nl} + (\partial_k B^{rn}_i) G^j_{nl} - (\partial_k B^{rj}_n) G^n_{il}) - \\
& - (\partial_i \partial_l B^{rj}_k - (\partial_i B^{rj}_{kn}) G^n_l + (\partial_i B^{nj}_k) G^r_{nl} + (\partial_i B^{rn}_k) G^j_{nl} - (\partial_i B^{rj}_n) G^n_{kl}) + \\
& + G^r_{km} (\partial_l B^{mj}_i - B^{mj}_{in} G^n_l + B^{nj}_i G^m_{nl} + B^{mn}_i G^j_{nl} - B^{mj}_n G^n_{il}) + \\
& + B^{mj}_i (\partial_l G^r_{km} - G^r_{kmn} G^n_l + G^n_{km} G^r_{nl} - G^r_{nm} G^n_{kl} - G^r_{kn} G^n_{ml}) + \\
& + G^m_i (\partial_l B^{rj}_{km} - B^{rj}_{kmn} G^n_l + B^{nj}_{km} G^r_{nl} + B^{rn}_{km} G^j_{nl} - B^{rj}_{nm} G^n_{kl} - B^{rj}_{kn} G^n_{ml}) + \\
& + B^{rj}_{km} (\partial_l G^m_i - G^m_{in} G^n_l + G^n_i G^m_{nl} - G^m_n G^n_{il}) - \\
& - G^m_k (\partial_l B^{rj}_{mi} - B^{rj}_{min} G^n_l + B^{nj}_{mi} G^r_{nl} + B^{rn}_{mi} G^j_{nl} - B^{rj}_{ni} G^n_{ml} - B^{rj}_{mn} G^n_{il}) - \\
& - B^{rj}_{mi} (\partial_l G^m_k - G^m_{kn} G^n_l + G^n_k G^m_{nl} - G^m_n G^n_{kl}) - \\
& - G^r_{mi} (\partial_l B^{mj}_k - B^{mj}_{kn} G^n_l + B^{nj}_k G^m_{nl} + B^{mn}_k G^j_{nl} - B^{mj}_n G^n_{kl}) - \\
& - B^{mj}_k (\partial_l G^r_{mi} - G^r_{min} G^n_l + G^n_{mi} G^r_{nl} - G^r_{ni} G^n_{ml} - G^r_{mn} G^n_{il})\} + \\
& + \sigma_j \sigma_s \{B^{rj}_{km} (\partial_l B^{ms}_i - B^{ms}_{in} G^n_l + B^{ns}_i G^m_{nl} + B^{mn}_i G^s_{nl} - B^{ms}_n G^n_{il}) + \\
& + B^{ms}_i (\partial_l B^{rj}_{km} - B^{rj}_{kmn} G^n_l + B^{nj}_{km} G^r_{nl} + B^{rn}_{km} G^j_{nl} - B^{rj}_{nm} G^n_{kl} - B^{rj}_{kn} G^n_{ml}) - \\
& - B^{rj}_{mi} (\partial_l B^{ms}_k - B^{ms}_{kn} G^n_l + B^{ns}_k G^m_{nl} + B^{mn}_k G^s_{nl} - B^{ms}_n G^n_{kl}) - \\
& - B^{ms}_k (\partial_l B^{rj}_{mi} - B^{rj}_{min} G^n_l + B^{nj}_{mi} G^r_{nl} + B^{rn}_{mi} G^j_{nl} - B^{rj}_{ni} G^n_{ml} - B^{rj}_{mn} G^n_{il})\} + \\
& + \{(B^{rj}_{km} B^{ms}_i - B^{rj}_{mi} B^{ms}_k\} \{\sigma_j (\partial_l \sigma_s - \sigma_n G^n_{sl}) + \sigma_s (\partial_l \sigma_j - \sigma_n G^n_{jl})\}] .
\end{aligned}
$$

With the help of equation (1.12) by cyclic interchange of indices i, l and k we obtain the result (1.1).

Again with the help of equations (1.11) and (1.12) we obtain the result (1.2).

THEOREM 2. *Under the conformal transformation we have:*

(2.1)
$$(\dot\partial_h \bar Q_k - \dot\partial_k \bar Q_h) = \frac{2}{(n-1)} [(\partial_{[h} G^i_{k]i} + G^i_{im[h} G^m_{k]} - G^m_i G^i_{m[kh]} -$$
$$- G^m_{i[h} G^i_{k]m}) - \sigma_j (\partial_{[k} B^{ij}_{h]i} + G^i_{im[h} B^{mj}_{k]} + B^{ij}_{im[h} G^m_{k]} -$$
$$- B^{mj}_i G^i_{m[kh]} - B^{mj}_{i[h} G^i_{k]m} - G^m_{i[h} B^{ij}_{k]m} - G^m_i B^{ij}_{m[kh]}) +$$
$$+ \sigma_j \sigma_s (B^{ij}_{im[h} B^{ms}_{k]} - B^{ms}_i B^{is}_{m[kh]} - B^{ms}_{i[h} B^{ij}_{k]m})],$$

and

(2.2)
$$\bar Q_{k(h)} - \bar Q_{h(k)} = -\frac{2}{(n-1)} [\{(\partial_i \partial_{[h} G^i_{k]} - (\partial_i G^i_{l[k}) G^l_{h]} + (\partial_i G^l_{[k}) G^i_{h]l}) -$$
$$- ((\partial_{[h} G^i_{il}) G^l_{k]} + (\partial_{[k} G^l_i) G^i_{h]l} - (\partial_{[k} G^i_l) G^l_{h]}) +$$
$$+ G^m_{[k} (\partial_{h]} G^i_{im} - G^l_{h]} G^i_{iml} + G^i_{h]l} G^l_{im} - G^l_{h]i} G^i_{lm} - G^l_{h]m} G^i_{il}) +$$
$$+ G^i_{im} (\partial_{[h} G^m_{k]} - 2 G^l_{[h} G^m_{k]l}) - G^m_i (\partial_{[h} G^r_{k]m} - G^l_{[h} G^r_{(m)k]l} + 2 G^l_{m[k} G^r_{h]l}) -$$
$$- G^r_{m[k} (\partial_{h]} G^m_i - G^l_{h]} G^m_{il} + G^m_{h]l} G^l_i - G^l_{h]i} G^m_l)\} - \{(\partial_{[h} \sigma_j - \sigma_l G^l_{j[h})\} \cdot$$
$$\cdot \{(\partial_i B^{ij}_{k]} - \partial_{k]} B^{ij}_i + B^{mj}_{k]} G^i_{im} + G^m_{k]} B^{ij}_{im} - G^i_{k]m} B^{mj}_i - G^m_i B^{ij}_{k]m})\} -$$
$$- \sigma_j \{(\partial_i \partial_{[h} B^{ij}_{k]} - (\partial_i B^{ij}_{l[k}) G^l_{h]} + (\partial_i B^{lj}_{[k}) G^i_{h]l} + (\partial_i B^{il}_{[k}) G^j_{h]l}) -$$
$$- ((\partial_{[h} B^{ij}_{(il)}) G^l_{k]} + (\partial_{[k} B^{lj}_{(i)}) G^i_{h]l} + (\partial_{[k} B^{il}_{(i)}) G^j_{h]l} + (\partial_{[k} B^{ij}_{(l)}) G^l_{h]i}) +$$
$$+ G^i_{im} (\partial_{[h} B^{mj}_{k]} - B^{mj}_{l[k} G^l_{h]} + B^{lj}_{[k} G^m_{h]l} + B^{ml}_{[k} G^j_{h]l}) +$$
$$+ B^{mj}_{[k} (\partial_{h]} G^i_{im} - G^l_{h]} G^i_{iml} + G^i_{h]l} G^l_{im} - G^l_{h]i} G^i_{lm} - G^l_{h]m} G^i_{il}) +$$
$$+ G^m_{[k} (\partial_{h]} B^{ij}_{im} - G^l_{h]} B^{ij}_{iml} + G^i_{h]l} B^{lj}_{im} + G^j_{h]l} B^{il}_{im} - G^l_{h]i} B^{ij}_{lm} - B^{ij}_{il} G^l_{h]m}) +$$
$$+ B^{ij}_{im} (\partial_{[h} G^m_{k]} - 2 G^m_{l[k} G^l_{h]}) -$$
$$- G^i_{m[k} (\partial_{h]} B^{mj}_i - G^l_{h]} B^{mj}_{li} + G^m_{h]l} B^{lj}_i + G^j_{h]l} B^{ml}_i - G^l_{h]i} B^{mj}_l) -$$
$$- B^{mj}_i (\partial_{[h} G^i_{k]m} - G^l_{[h} G^i_{(m)k]l} + 2 G^l_{m[k} G^i_{h]l}) -$$
$$- G^m_i (\partial_{[h} B^{ij}_{k]m} - G^l_{[h} B^{ij}_{(m)k]l} + B^{lj}_{m[k} G^i_{h]l} + B^{il}_{m[k} G^j_{h]l} - B^{ij}_{l[k} G^l_{h]m}) -$$
$$- B^{ij}_{m[k} (\partial_{h]} G^m_i - G^l_{h]} G^m_{il} + G^m_{h]l} G^l_i - G^l_{h]i} G^m_l)\} +$$
$$+ \sigma_j \sigma_s \{B^{ij}_{im} (\partial_{[h} B^{ms}_{k]} - B^{ms}_{l[k} G^l_{h]} + B^{ls}_{[k} G^m_{h]l} + B^{ml}_{[k} G^s_{h]l}) +$$
$$+ B^{ms}_{[k} (\partial_{h]} B^{ij}_{im} - G^l_{h]} B^{ij}_{iml} + G^i_{h]l} B^{lj}_{im} + G^j_{h]l} B^{il}_{im} - G^l_{h]i} B^{ij}_{lm} - G^l_{h]m} B^{ij}_{il}) -$$
$$- B^{ij}_{m[k} (\partial_{h]} B^{ms}_i - G^l_{h]} B^{ms}_{il} + G^m_{h]l} B^{ls}_i - G^s_{h]l} B^{ml}_i - G^l_{h]i} B^{ms}_l) -$$
$$- B^{ms}_i (\partial_{[h} B^{ij}_{k]m} - G^l_{[h} B^{ij}_{(m)k]l} + B^{lj}_{m[k} G^i_{h]l} + B^{il}_{m[k} G^j_{h]l} - B^{ij}_{l[k} G^l_{h]m})\} +$$
$$+ \{B^{ij}_{im} B^{ms}_{[k} - B^{ms}_i B^{is}_{m[k}\} \{\sigma_j (\partial_{h]} \sigma_s - \sigma_l G^l_{h]s}) + \sigma_s (\partial_{h]} \sigma_j - \sigma_l G^l_{h]j})\}].$$

Proof. Let us suppose that [1]

$$Q_k = P_{(k)} + P(\dot{\partial}_k P) \tag{2.3}$$

where $P(x, \dot{x})$ is positively homogeneous of first degree in its directional arguments and the projective transformation is defined by

$$\overline{G}^i(x, \dot{x}) = G^i(x, \dot{x}) - P(x, \dot{x})\, \dot{x}^i. \tag{2.4}$$

Therefore Q_k is also positively homogeneous of first degree in its directional arguments, with the help of equations (2.3), (2.4) and

$$(\dot{\partial}_k T)_{(h)} - (\dot{\partial}_k T_{(h)}) = 0 \tag{2.5}$$

$$\overline{H}^j_{hk} = H^j_{hk} - \delta^j_h (P_{(k)} + P(\dot{\partial}_k P)) + \delta^j_k (P_{(h)} + P(\dot{\partial}_h P)) - \\ - \dot{x}^j [(\dot{\partial}_h P)_{(k)} - (\dot{\partial}_k P)_{(h)}] \tag{2.6}$$

and by putting $\overline{H}^j_{hk} = 0$, we have,

$$H^j_{hk} = \delta^j_h Q_k + \dot{x}^j (\dot{\partial}_h Q_k - \dot{\partial}_k Q_h). \tag{2.7}$$

If we contract equation (2.7) in the indices j and k we have,

$$H^r_{hr} = H_h = - nQ_h - (\dot{\partial}_h Q_r)\, \dot{x}^r \tag{2.8}$$

but,
$$H_{ij} = H^h_{ijh}.$$

Therefore from equation (2.8) we get,

$$H_{kh} = - n\dot{\partial}_k Q_r\, \dot{x}^r + Q_k. \tag{2.9}$$

By eliminating the directional derivative of Q_r between equations (2.8) and (2.9) we have, [1]

$$Q_k = - \frac{1}{(n^2 - 1)} (nH_k + H_{kr}\, \dot{x}^r). \tag{2.10}$$

Under the conformal transformation defined in equations (1.1 h), (1.1 i) and (1.1 k) we have from equations (1.1 m) and (1.1 n),

$$\overline{H}_k = \overline{H}^i_{ki} = [(\partial_i G^i_k - \partial_k G^i_i) - \sigma_j (\partial_i B^{ij}_k - \partial_k B^{ij}_i) + \\ + (G^i_{im} G^m_k - G^i_{mk} G^m_i) - \sigma_j (G^i_{im} B^{mj}_k + G^m_k B^{ij}_{im} - G^i_{mk} B^{mj}_i - G^m_i B^{ij}_{mk}) + \\ + \sigma_j \sigma_s (B^{ij}_{im} B^{ms}_k - B^{ij}_{mk} B^{ms}_i)], \tag{2.11}$$

(3) Here we have made use of $\dot{\partial}_i G^r_k = G^r_{ki}$ and $\dot{\partial}_i B^{rm}_j = B^{rm}_{ji}$.

and

$$(2.12)\quad \overline{H}_{kr} = [(\partial_i G^i_{kr} - \partial_k G^i_{ir}) - \sigma_j (\partial_i B^{ij}_{kr} - \partial_k B^{ij}_{ir}) +$$
$$+ (G^i_{imr} G^m_k + G^i_{im} G^m_{kr} - G^m_i G^i_{mkr} - G^m_{ir} G^i_{mk}) -$$
$$- \sigma_j (G^i_{imr} B^{mj}_k + B^{mj}_{kr} G^i_{im} + G^m_{kr} B^{ij}_{im} + G^m_k B^{ij}_{imr} -$$
$$- G^i_{mkr} B^{mj}_i - B^{mj}_{ir} G^i_{mk} - B^{ij}_{mk} G^m_{ir} - G^m_i B^{ij}_{mkr}) +$$
$$+ \sigma_j \sigma_s (B^{ij}_{imr} B^{ms}_k + B^{ms}_{kr} B^{ij}_{im} - B^{ij}_{mkr} B^{ms}_i - B^{ms}_{ir} B^{ij}_{mk})].$$

But we also know that

$$(2.13)\quad G^i_{kr}(x, \dot{x})\, \dot{x}^r = G^i_k(x, \dot{x})$$

$$(2.14)\quad G^i_{khr}(x, \dot{x})\, \dot{x}^r = 0$$

$$(2.15)\quad G^i_{kh} = G^i_{hk}$$

and $B^{hk}(x, \dot{x})$ are positively homogeneous of second degree in its directional arguments.

With the help of equations (2.10), (2.11), (2.12), (2.13) and (2.14) we have,

$$(2.16)\quad \overline{Q}_k = -\frac{1}{(n-1)} [(\partial_i G^i_k - \partial_k G^i_i + G^i_{im} G^m_k - G^i_{mk} G^m_i) -$$
$$- \sigma_j (\partial_i B^{ij}_k - \partial_k B^{ij}_i + (G^i_{im} B^{mj}_k + G^m_k B^{ij}_{im} - G^i_{mk} B^{mj}_i - G^m_i B^{ij}_{mk}) +$$
$$+ \sigma_j \sigma_s (B^{ij}_{im} B^{ms}_k - B^{ij}_{mk} B^{ms}_i)].$$

Differentiating partially equation (2.16) with respect to $\dot{x}^h$ we get,

$$(2.17)\quad (\dot{\partial}_h \overline{Q}_k) = -\frac{1}{(n-1)} [(\partial_i G^i_{kh} - \partial_k G^i_{ih} + G^i_{imh} G^m_k +$$
$$+ G^i_{im} G^m_{kh} - G^i_{mkh} G^m_i - G^m_{ih} G^i_{mk}) -$$
$$- \sigma_j (\partial_i B^{ij}_{kh} - \partial_k B^{ij}_{ih} + G^i_{imh} B^{mj}_k + B^{mj}_{kh} G^i_{im} + G^m_{kh} B^{ij}_{im} +$$
$$+ G^m_k B^{ij}_{imh} - G^i_{mkh} B^{mj}_i - B^{mj}_{ih} G^i_{mk} - G^m_{ih} B^{ij}_{mk} - G^m_i B^{ij}_{mkh}) +$$
$$+ \sigma_j \sigma_s (B^{ij}_{imh} B^{ms}_k + B^{ij}_{im} B^{ms}_{kh} - B^{ij}_{mkh} B^{ms}_i - B^{ms}_{ih} B^{ij}_{mk})]$$

and also differentiating covariantly equation (2.16) with respect to x^h in the sense of Berwald we get,

$$(2.18)\quad \overline{Q}_{k(h)} = -\frac{1}{(n-1)} [\{(\partial_i \partial_h G^i_k - (\partial_i G^i_{kl}) G^l_h + (\partial_i G^l_k) G^i_{lh} - (\partial_i G^i_l) G^l_{kh}) -$$
$$- (\partial_k \partial_h G^i_i - (\partial_k G^i_{il}) G^l_h + (\partial_k G^l_i) G^i_{lh} - (\partial_k G^i_l) G^l_{ih}) +$$
$$+ G^m_k (\partial_h G^i_{im} - G^i_{iml} G^l_h + G^l_{im} G^i_{lh} - G^i_{lm} G^l_{ih} - G^i_{il} G^l_{mh}) +$$
$$+ G^i_{im} (\partial_h G^m_k - G^m_{kl} G^l_h + G^l_k G^m_{lh} - G^m_l G^l_{kh}) -$$

$$
\begin{aligned}
& - G_i^m (\partial_h G^i_{mk} - G^i_{mkl} G^l_h + G^l_{mk} G^i_{lh} - G^i_{lk} G^l_{mh} - G^i_{ml} G^l_{kh}) - \\
& - G^i_{mk} (\partial_h G^m_i - G^m_{il} G^l_h + G^l_i G^m_{lh} - G^m_l G^l_{ih})\} - \{(\partial_h \sigma_j - \sigma_l G^l_{jh})\} \cdot \\
& \cdot \{\partial_i B^{ij}_k - \partial_k B^{ij}_i + G^i_{im} B^{mj}_k + G^m_k B^{ij}_{im} - G^i_{mk} B^{mj}_i - G^m_i B^{ij}_{mk})\} - \\
& - \sigma_j \{(\partial_i \partial_h B^{ij}_k - (\partial_i B^{ij}_{kl}) G^l_h + (\partial_i B^{lj}_k) G^i_{lh} + (\partial_i B^{il}_k) G^j_{lh} - (\partial_i B^{ij}_l) G^l_{kh}) - \\
& - (\partial_k \partial_h B^{ij}_i - (\partial_k B^{ij}_{il}) G^l_h + (\partial_k B^{lj}_i) G^i_{lh} + (\partial_k B^{il}_i) G^j_{lh} - (\partial_k B^{ij}_l) G^l_{ih}) + \\
& + G^i_{im} (\partial_h B^{mj}_k - B^{mj}_{kl} G^l_h + B^{lj}_k G^m_{lh} + B^{ml}_k G^j_{lh} - B^{mj}_l G^l_{kh}) + \\
& + B^{mj}_k (\partial_h G^i_{im} - G^i_{iml} G^l_h + G^l_{im} G^i_{lh} - G^i_{lm} G^l_{ih} - G^i_{il} G^l_{mh}) + \\
& + G^m_k (\partial_h B^{ij}_{im} - B^{ij}_{iml} G^l_h + B^{lj}_{im} G^i_{lh} + B^{il}_{im} G^j_{lh} - B^{ij}_{lm} G^l_{ih} - B^{ij}_{il} G^l_{mh}) + \\
& + B^{ij}_{im} (\partial_l G^m_k - G^m_{kl} G^l_h + G^l_k G^m_{lh} - G^m_l G^l_{kh}) - \\
& - G^i_{mk} (\partial_h B^{mj}_i - B^{mj}_{il} G^l_h + B^{lj}_i G^m_{lh} + B^{ml}_i G^j_{lh} - B^{mj}_l G^l_{ih}) - \\
& - B^{mj}_i (\partial_h G^i_{mk} - G^i_{mkl} G^l_h + G^l_{mk} G^i_{lh} - G^i_{lk} G^l_{mh} - G^i_{ml} G^l_{kh}) - \\
& - G^m_i (\partial_h B^{ij}_{mk} - B^{ij}_{mkl} G^l_h + B^{lj}_{mk} G^i_{lh} + B^{il}_{mk} G^j_{lh} - B^{ij}_{lk} G^l_{mh} - B^{ij}_{ml} G^l_{kh}) - \\
& - B^{ij}_{mk} (\partial_h G^m_i - G^m_{il} G^l_h + G^l_i G^m_{lh} - G^m_l G^l_{ih})\} + \\
& + \sigma_j \sigma_s \{B^{ij}_{im} (\partial_h B^{ms}_k - B^{ms}_{kl} G^l_h + B^{ls}_k G^m_{lh} + B^{ml}_k G^s_{lh} - B^{ms}_l G^l_{kh}) + \\
& + B^{ms}_k (\partial_h B^{ij}_{im} - B^{ij}_{iml} G^l_h + B^{lj}_{im} G^i_{lh} + B^{il}_{im} G^j_{lh} - B^{ij}_{lm} G^l_{ih} - B^{ij}_{il} G^l_{mh}) - \\
& - B^{ij}_{mk} (\partial_h B^{ms}_i - B^{ms}_{il} G^l_h + B^{ls}_i G^m_{lh} + B^{ml}_i G^s_{lh} - B^{ms}_l G^l_{ih}) - \\
& - B^{ms}_i (\partial_h B^{ij}_{mk} - B^{ij}_{mkl} G^l_h + B^{lj}_{mk} G^i_{lh} + B^{il}_{mk} G^j_{lh} - B^{ij}_{lk} G^l_{mh} - B^{ij}_{ml} G^l_{kh})\} + \\
& + \{B^{ij}_{im} B^{ms}_k - B^{is}_{mk} B^{ms}_i\} \{\sigma_j (\partial_h \sigma_s - \sigma_l G^l_{sh}) + \sigma_s (\partial_h \sigma_j - \sigma_l G^l_{jh})\}] .
\end{aligned}
$$

With the help of equations (2.17) and (2.15) we get the result (2.1), again with the help of equations (2.15) and (2.18) we get the result (2.2).

References

[1] RUND H., *The Differential Geometry of Finsler Spaces*, Springer–Verlag, Berlin 1959.

[2] PANDE H. D., *Various projective invariants in a Conformal Finsler Space*, « Atti della Accad. Naz. Lincei Rendiconti », 46 (3) 349–353 (1968).

[3] HOMBU H., *Konforme Invarianten in Finslerschen Raume*, « J. Fac. Sc. Hokkaido Univ. Ser. I. Math », 2 157–168 (1934).

[4] MISHRA R. S., *A cource in tensors with application to Riemannian Geometry Pothishala Private limited*, Allahabad, India (1965).

[5] EISENHART L. P., *A treatise on the Differential Geometry of Curves and surfaces*, Ginn. co. New York (1909).