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A note on the minimum property of $\overline{R(A)}$ for a monotone mapping in a real Hilbert space

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Analisi funzionale. — *A note on the minimum property of* $\overline{R(A)}$ *for a monotone mapping in a real Hilbert space.* Nota (*) di
ADRIAN CORDUNEANU, presentata dal Socio G. SANSONE.

RIASSUNTO. — Lo scopo di questa Nota è di dare una generalizzazione di un risultato di Pazy relativo all'asintotico comportamento di una contrazione in uno spazio di Hilbert e di fare qualche osservazione circa il semigruppo di contrazione generato da una certa monotona trasformazione.

1. INTRODUCTION

Let H be a real Hilbert space with the scalar product $(\cdot,\cdot)$ and the identity operator I and let A be a subset of $H \times H$. For such a subset "the domain of definition" is given by $D(A) = \{x ; [x, y] \in A \text{ for some } y \in H\}$ and "the range" is given by $R(A) = \{y ; [x, y] \in A \text{ for some } x \in H\}$. A subset $A \subset H \times H$ may be considered as a multivalued mapping defined on $D(A) \subset H$ with the values in $R(A) \subset H$, defining $Ax = \{y ; [x, y] \in A\}$ for $x \in D(A)$. For a real λ, we define $\lambda A = \{[x, \lambda y] ; [x, y] \in A\}$, and for $A, B \subset H \times H$ we define $A + B = \{[x, y + z] ; [x, y] \in A \text{ and } [x, z] \in B\}$.

We say that A is monotone if for every $[x_i, y_i] \in A$, $i = 1, 2$, we have $(y_1 - y_2, x_1 - x_2) \geq 0$. A is said to be maximal monotone if it is monotone and there is no monotone set $\tilde{A}$ such that $A \subset \tilde{A}$, $A \neq \tilde{A}$.

A subset $A \subset H \times H$ is said to be closed if $[x_n, y_n] \in A$, $x_n \to x$ and $y_n \to y$ imply that $[x, y] \in A$.

Following A. Pazy [5], we say that a closed set $K \subset H$ has the minimum property if the element of minimum norm in $\overline{\text{conv}\, K}$ belongs to K. (Here, $\overline{\text{conv}\, K}$ denotes the closure of the convex hull of K).

We shall use the following well known:

LEMMA 1 [5]. *Let $C \subset H$ be a closed convex set and let v be the minimal element of C. If $u_n \in C$ and $|u_n| \to |v|$, then $u_n \to v$.*

Another result employed in this paper is an extension of a theorem due to H. Brézis and A. Pazy [1], namely:

LEMMA 2. *Let $A \subset H \times H$ be a closed monotone set. If*

$$R(I + \lambda A) \supset \overline{\text{conv}\, D(A)} \qquad \textit{for every} \quad \lambda > 0$$

then

a) $\overline{D(A)}$ *is convex;*
b) *For every $x \in D(A)$, Ax has an element of minimum norm denoted by $A^0 x$.*
c) *$-A^0$ is the generator of a semigroup of contractions on $\overline{D(A)}$.*
d) *A has a unique extension to a maximal monotone set $\tilde{A}$ satisfying $D(\tilde{A}) = D(A)$ and $\tilde{A}^0 = A^0$.*

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We shall also use the following result due to M. G. Crandall (unpublished; see H. Brézis [2]).

LEMMA 3. *Let* $A \subset H \times H$ *be a maximal monotone set. Let be* $f_\infty \in H$ *and* $f(t)$ *such that* $f(t) - f_\infty \in L^1(0, +\infty; H)$. *Then for every weak solution* u *of the equation* $du/dt + Au \ni f$, *we have* $\lim_{t\to\infty} u(t)/t = -v$, *where* v *is the minimal element in* $\overline{R(A)} - f$.

2. THEOREM 1. *Let* $A \subset H \times H$ *be a closed monotone set and*

$$R(I + \lambda A) \supset \overline{\operatorname{conv} D(A)} \qquad \text{for every} \quad \lambda > 0.$$

Then $\overline{R(A)}$ *has the minimum property.*

Proof. By the Lemma 2 quoted above, A has a maximal extension $\tilde{A}$ satisfying $D(\tilde{A}) = D(A)$ and $\tilde{A}^0 = A^0$; on the other hand it is well known that $\overline{R(\tilde{A})}$ is a convex set. Thus $\overline{R(\tilde{A})} = \overline{\operatorname{conv} R(\tilde{A})} \supset \overline{\operatorname{conv} R(A)} \supset \overline{R(A)}$. Let v be the minimal element in $\overline{R(A)}$; thus, there is $x_n \in D(A)$ and $v_n \in Ax_n$, $n = 1, 2, 3, \cdots$, such that $v_n \to v$. Because $|v_n| \geq |\tilde{A}^0 x_n| = |A^0 x_n| \geq |v|$ follows that $|A^0 x_n| \to |v|$ and by the Lemma 1 we obtain $A^0 x_n \to v$, i.e. $v \in \overline{R(A)}$. Obviously v is the minimal element in $\overline{\operatorname{conv} R(A)}$, thereby the theorem is proved.

As a corollary we obtain a theorem of Pazy [5] about the asymptotic behaviour of contractions. A function T defined on $D(T) \subset H$ with the values in H is called a contraction if $|Tx - Ty| \leq |x - y|$, $\forall x, y \in D(T)$. The function $A = I - T$ defined on $D(A) = D(T)$ is obviously monotone and if $D(A)$ is closed A is also closed. All the conditions required in the Theorem 1 are satisfied if $T : C \to C$ is a contraction, where C is convex and closed. On the other hand, $\{S(t)\}$, $t \geq 0$ being the semigroup of contractions generated by $-A$, it is known ([1], Lemma 3.2) the inequality $|S(n)x - T^n x| \leq \sqrt{n}\,|Ax|$, $\forall x \in D(A)$ and $n = 1, 2, 3, \cdots$. Since $S(n)x/n \to -v$ when $n \to \infty$, where v is the minimal element in $\overline{R(I - T)}$, we obtain that $T^n x/n \to -v$ when $n \to \infty$, $\forall x \in C$ and so it was proved the following result established by A. Pazy ([5], Theorem 2 and Lemma 5):

COROLLARY 1. *Let* $C \subset H$ *be a closed convex set and let* $T : C \to C$ *be a contraction. Then* $T^n x/n \to -v$ *when* $n \to \infty$, $\forall x \in C$, *where* v *is the minimal element in* $\overline{R(I - T)}$.

COROLLARY 2. *Suppose that the conditions of Theorem 1 are satisfied and* $D(A)$ *is bounded and weakly closed. Then* $0 \in R(A)$.

Proof. Inasmuch as $D(A)$ is bounded, it follows that $|S(t)x| \leq M_x$ for $t \geq 0$, $\forall x \in D(A)$, where $\{S(t)\}$, $t \geq 0$ is the semigroup of contractions on $\overline{D(A)}$ generated by A. Thus $S(t)x/t \to 0$ when $t \to \infty$ and consequently $0 \in \overline{R(A)}$ i.e. there is $x_n \in D(A)$ and $y_n \in Ax_n$, $n = 1, 2, 3, \cdots$, such that $y_n \to 0$. We may assume $x_n \rightharpoonup \xi \in D(A)$, the arrow $\rightharpoonup$ denoting the weak convergence in H. Because $(y_n - y, x_n - x) \geq 0$, $\forall [x, y] \in A$ and $n = 1, 2, 3, \cdots$, it follows $(y, \xi - x) \leq 0$, $\forall [x, y] \in A$. Taking x_0 the unique solution of the equation $\xi \in (I + \lambda A) x_0$, $\lambda > 0$ being fixed, and $y_0 \in A x_0$ such that $\xi = x_0 + \lambda y_0$, we obtain $(y_0, \lambda y_0) \leq 0$ which implies $y_0 = 0$, i.e. $0 \in R(A)$.

COROLLARY 3 (Browder's fixed point theorem). *Let* $C \subset H$ *be closed, convex and bounded. If* $T : C \to C$ *is a contraction, then* T *has a fixed point in* C.

Proof. It suffices to take $A = I - T$. By the Corollary 2, $0 \in R(A)$ i.e. $\exists x_0 \in C$ such that $x_0 = T x_0$.

3. In this section we study the behaviour of the semigroup $\{S(t)\}$, $t \geq 0$ generated by the closed monotone set A, possessing the property required in the Lemma 2.

THEOREM 2. *Suppose that the conditions of the Theorem 1 are satisfied. Then*

a) $0 \in R(A) \Longleftrightarrow |S(t)x|$ *is bounded for* $t \geq 0$, $\forall x \in D(A)$.

b) $0 \in \overline{R(A)} - R(A) \Longleftrightarrow |S(t)x|$ *is unbounded for* $t \geq 0$, $\forall x \in D(A)$ *and* $\lim_{t\to\infty} |S(t)x|/t = 0$, $\forall x \in D(A)$.

c) $0 \notin \overline{R(A)} \Longleftrightarrow \lim_{t\to\infty} |S(t)x|/t = \alpha > 0$, $\forall x \in D(A)$.

Proof. *a)* If $0 \in A x_0$, $S(t) x_0 \equiv x_0$ for $t \geq 0$. Let $x \in D(A)$; then $|S(t)x| \leq |S(t)x_0| + |S(t)x - S(t)x_0| \leq |x_0| + |x - x_0|$, $\forall t \geq 0$. Conversely, let $|S(t)x|$ be bounded for $t \geq 0$, $\forall x \in D(A)$. $\{S(t)\}$, $t \geq 0$ may be considered as a bounded contraction semigroup on the convex set $\overline{D(A)}$ and then follows from ([3], Corollary 5.1) that $0 \in R(A)$. (In fact, $0 \in R(A^0)$).

b) Taking into account *a)* and the Lemma 3, the Proof is immediate.

c) Taking into account *a)*, *b)* and the Lemma 3, the proof is very easy.

COROLLARY 1. *Let* $C \subset H$ *be a closed convex set and let* $T : C \to C$ *be a contraction. Consider the equation*

$$\text{(E)} \qquad du/dt + (I - T) u = 0 \quad , \quad u(0) = x \in C .$$

Then

a) *All solution of equations* (E) *are bounded* $\Longleftrightarrow |T^n x|$ *is bounded*, $\forall x \in C$.

b) *All solutions of equation* (E) *are unbounded and* $\lim_{t\to\infty} |u(t)|/t = 0$, $\forall x \in C \Longleftrightarrow |T^n x|$ *is unbounded and* $\lim_{n\to\infty} |T^n x|/n = 0$, $\forall x \in C$.

c) *All solutions of equation* (E) *satisfy the condition*

$$\lim_{t\to\infty} |u(t)|/t = \alpha > 0 \Longleftrightarrow \lim_{n\to\infty} |T^n x|/n = \alpha, \qquad \forall x \in C.$$

Proof. It is immediate if we take into account the Theorem 2 and a result of Pazy ([5], Corollary 6) concerning the behaviour of the sequence $|T^n x|/n$.

4. In this section, we give a theorem of asymptotic stability analogous to that of R. H. Martin ([4], Theorem 1).

THEOREM 3. *Assume that* $A : D(A) \to H$ *is a function such that*:

(i) *The conditions of Theorem 1 are satisfied.*

(ii) $(Ax - Ay, x - y) \geq \rho(r) |x - y|^2$ if $|x|, |y| \leq r$, where $\rho = \rho(r)$ *is a positive function defined on* $[0, +\infty)$.

(iii) *There is* $x_c \in D(A)$, *with* $Ax_c = 0$.

Then for every solution $u = u(t)$ *of the equation* $du/dt + Au = 0$. $u(0) = x \in D(A)$, *it follows* $\lim_{t \to \infty} u(t) = x_c$.

Proof. Every solution $u(t) = S(t)x$ of the above equation is bounded: $|S(t)x| \leq |S(t)x_c| + |S(t)x - S(t)x_c| \leq |x_c| + |x - x_c|$ for $t \geq 0$. If we put $p(t) = |u(t) - u_1(t)|$ for two different solutions of the given equation and we denote by $p'_+(t)$ the right derivative of $p(t)$, it follows $p'_+(t) = -(Au(t) - Au_1(t), u(t) - u_1(t))/|u(t) - u_1(t)| \leq -\rho(r_0)p(t)$, where r_0 is a positive number such that $|u(t)|, |u_1(t)| \leq r_0$ for $t \geq 0$. Thus we have $p(t) \leq p(0) \exp(-\rho(r_0)t)$ for $t \geq 0$. Taking $u_1(t) = S(t)x_c \equiv x_c$ and $r_0 = |x_c| + |x - x_c|$, we finally obtain that

$$|u(t) - x_c| \leq |x - x_c| \cdot \exp(-\rho(|x_c| + |x - x_c|)t) \quad \text{for} \quad t \geq 0 .$$

Remark. The solutions $u(t) = S(t)x$ of the equation $du/dt + Au = 0$, $u(0) = x$ considered in this paper are strong solutions, i.e. are continuous and almost everywhere derivable.

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