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Fred Gross, Chung-Chun Yang

On the growth of the meromorphic solutions of certain functional—differential equations

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Analisi funzionale. — *On the growth of the meromorphic solutions of certain functional–differential equations* (*). Nota (**) di Fred Gross e Chung–Chun Yang, presentata dal Socio G. Sansone.

Riassunto. — Si considera l'accrescimento di una funzione meromorfa, soluzione dell'equazione

$$P(u(\lambda_1 z), u(\lambda_2 z), \cdots, u(\lambda_l z)) = h(u^{(m)}(z))$$

dove m è un intiero ≥ 0.

$P(w_1(z), \cdots, w_l(z))$ è un polinomio delle funzioni $w_1(z), \cdots, w_l(z)$ e delle loro derivate avente come coefficienti polinomi in z, h è una data funzione meromorfa d'ordine zero e λ_i, $i = 1, 2, \cdots, l$ sono costanti in valore assoluto > 1.

Let $g(z)$ denote a non-constant meromorphic function. Then, as usual, the Nevanlinna characteristic function $T(r, g)$ is used to measure the growth rate of g and which has many properties in analogy with the logarithm of the maximum modulus function of an entire function. In particular, $T(r, g)$ is a real-valued, continuous non-decreasing and unbounded function defined for $r > r_0 > 0$. The order ρ_g and lower order μ_g of g are defined as

$$\rho_g = \overline{\lim_{r\to\infty}} \frac{\log T(r, g)}{\log r} \quad \text{and} \quad \mu_g = \underline{\lim_{r\to\infty}} \frac{\log T(r, g)}{\log r}.$$

In this paper we are primarily interested in the investigation of the growth rate of the meromorphic functions $u(z)$ which are solutions of the equation of the form:

$$P(u(\lambda_1 z), u(\lambda_2 z), \cdots, u(\lambda_l z)) = h(u^{(m)}(z)), \tag{1}$$

where $P(w_1(z), w_2(z), \cdots, w_l(z))$ denotes a polynomial in l functions $w_1(z), \cdots, w_l(z)$ and their derivatives with polynomials in z as the coefficients, h is a given meromorphic function, $m = 0$ or any positive integer $(u^{(0)}(z) \equiv u(z))$, and λ_i are constants with $|\lambda_i| > 1$.

Our study of the above equation is motivated by the following equation which is a special case of equation (1)

$$u(\lambda z) = h(u(z)). \tag{2}$$

Equation (2) is called Poincaré equation and has been investigated by many authors. We here refer the reader to a book of Kuczma's [8, p. 141] for references.

(*) The content is based on a talk delivered by the second author at the Conference on Ordinary Differential Equations at Oberwolfach, Federal Republic of Germany, March 23, 1972.

(**) Pervenuta all'Accademia il 3 luglio 1972.

Before proceeding further we would like to point out that for certain functions h, there exist solutions to equation (2). For example, when $h(z) = e^z$, a solution $u(z)$ can be exhibited for appropriate constants c. Baker [1] showed that for a constant c with $|c| > 1$, there exists an entire function f satisfying the equation:

$$f(cz) = \exp f(z).$$

With $\lambda = c$, $u = f$ is a solution of (2). It is pointed out [1] that the growth of $f(z)$ is faster than that of any $\exp_n(z)$ (the n-th iterate of $\exp(z)$).

As a second example let $h(z) = z^2$, $\lambda = 2$. In this case $u(z) = e^z$ is a solution of equation (2).

It is easily seen, by virtue of a result of Edrei and Fuchs [6] that when h is a meromorphic function of positive order then every transcendental entire solution of equation (2) has infinite order. Therefore we shall only treat the case when the given meromorphic function h is of zero order.

In the sequel, we shall show that when h is transcendental, then any entire solution $u(z)$ of equation (2) is of infinite lower order. We shall also derive a more precise estimate for the growth of a meromorphic solution $u(z)$ when h is a rational function. Actually, we have

THEOREM 1. *Let $h(z)$ be a given non-rational meromorphic function of zero order. Then any entire function $u(z)$ which satisfies equation* (1), *has lower order equal to infinity.*

THEOREM 2. *Let $h(z)$ be a rational function of weight n $(n \geq 2)$. Suppose that $u(z)$ is a meromorphic function of order ρ $(0 \leq \rho \leq +\infty)$ which satisfies equation* (2) *with $|\lambda| > 1$. Then ρ must be finite and equal to $\log n / \log |\lambda|$. Moreover, u is of regular growth i.e. the lower order of $u(z)$ is equal to the order of $u(z)$.*

Remark. Theorem 2 is an extension of a special case of a result of Valiron [9, p. 46].

2. PRELIMINARY LEMMAS

Our theorems will readily follow from the following four Lemmas.

LEMMA 1. *Let g be transcendental entire function. Then*

(i)
$$\lim_{r \to \infty} \frac{T(r, f(g))}{T(r, g)} = \infty,$$

for any transcendental meromorphic function f (see e.g. [2]);

(ii)
$$\lim_{r \to \infty} \frac{T(r, R(g))}{T(r, g)} = n$$

for any rational function $R(z)$ of weight n (see e.g. [7]).

Lemma 2 will involve the notion of Polya peaks which we introduce first. Let $G(t)$ be a real valued, non-negative and unbounded function defined for $t \geq t_0 > 0$, and define the order and lower order of G respectively by

$$\rho = \overline{\lim_{t\to\infty}} \frac{\log G(t)}{\log t} \quad , \quad \mu = \underline{\lim_{t\to\infty}} \frac{\log G(t)}{\log t} .$$

Definition [3, 4, 5]. An increasing positive sequence $r_1, r_2, \cdots, r_m, \cdots$ is said to be a sequence of Polya peaks of order η for $G(t)$ $(0 \leq \eta < \infty)$ if it is possible to find a pair of associated sequences $\{a_m\}_{m=1}^{\infty}$, $\{A_m\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} A_m/r_m = +\infty \quad ; \quad \lim_{m\to\infty} r_m/a_m = \infty$$

and such that

$$G(t) \leq (1 + o(1)) (t/r_m)^{\eta} G(r_m) \; (m \to \infty , a_m \leq t \leq A_m) .$$

LEMMA 2. (*Existence theorem for Polya peaks* [4, 5]). *Let* $G(t)$ *be a real valued non-negative non-decreasing and unbounded function defined for* $t \geq t_0 > 0$, *having finite lower order* μ. *Then for each finite* η *satisfying* $\mu \leq \eta \leq \rho$ *there exists a sequence* $\{r_m\}$ *of Polya peaks, order* η, *of* $G(t)$.

LEMMA 3. *Let f be a non-constant meromorphic function. Suppose that there exists a constant* $\alpha > 1$ *such that the following estimate holds*:

$$\underline{\lim_{r\to\infty}} \frac{T(\alpha r, f)}{T(r, f)} = \infty . \tag{3}$$

Then the lower order of f must be infinite.

Proof. Suppose that $\mu_f < \infty$. Then there exists a sequence $\{r_m\}$ of Polya peaks of order μ_f for $T(r, f)$.

Hence, we would have

$$\underline{\lim_{m\to\infty}} \frac{T(\alpha r_m, f)}{T(r_m, f)} \leq (1 + o(1)) \, \alpha^{\mu_f} < +\infty . \tag{4}$$

This contradicts assumption (3), and Lemma 3 is thus proved.

LEMMA 4 [10, p. 25]. *Let f be a non-constant meromorphic function and* β *be any constant* > 1. *Then for* $r > r_0$

$$T(r, f') < K_1 T(\beta r, f) \tag{a}$$

and

$$T(r, f) < K_2 T(\beta r, f') , \tag{b}$$

where K_1 *and* K_2 *are two positive constants.*

3. PROOF OF THEOREM I.

We assume that $m = 0$. Suppose that u is a transcendental entire function satisfying equation (1). Then according to inequality (*a*) of Lemma 4, it is easy to show that there exist positive constants K and $\alpha > 1$ such that

$$\text{(5)} \qquad T(r, P(u(\lambda z), \cdots, u(\lambda_l z))) \leq KT(\alpha r, u)$$

for sufficiently large r.

From this and equation (1) we have

$$\text{(6)} \qquad KT(\alpha r, u) \geq T(r, h(u)) .$$

Hence, by Lemma 1 part (*i*)

$$\text{(7)} \qquad \underline{\lim}_{r\to\infty} \frac{T(\alpha r, u)}{T(r, u)} \geq \frac{1}{K} \underline{\lim}_{r\to\infty} \frac{T(r, h(u))}{T(r, u)} = \infty$$

our assertion follows from this and Lemma 3.

The proof for the case $m > 0$ is similar and will be omitted.

Proof of Theorem 2. Suppose that u is a transcendental meromorphic function satisfying equation (2) with h being a rational function of weight $n (n \geq 2)$. Then giving $\varepsilon > 0$ according to assertion (*ii*) of Lemma 1 we have (for $r > r_0 \geq 1$)

$$\text{(8)} \qquad T(|\lambda| r, u) = T(r, h(u)) \leq n(1 + \varepsilon) T(r, u) .$$

Thus

$$\text{(9)} \qquad T(|\lambda|^m r, u) \leq n^m (1 + \varepsilon)^m T(r, u) .$$

Now fix $r \geq r_0 \geq 1$. Then

$$\text{(10)} \qquad T(|\lambda|^m r_0, u) \leq n^m (1 + \varepsilon)^m T(r_0, u) .$$

Assume now that the order of $u (= \rho_u) = \rho$. Then there exists a sequence $\{r_n\}$ such that

$$\text{(11)} \qquad \lim_{n\to\infty} \frac{\log T(r_n, u)}{\log r_n} = \rho .$$

On the other hand if we choose for $i = 1, 2 \cdots$

$$\text{(12)} \qquad m_i = \frac{\log r_i}{\log |\lambda|} + 1$$

(where $[x]$ = greatest integer not exceeding x), then

$$\text{(13)} \qquad |\lambda|^{m_i} r_0 > r_i .$$

It follows from inequalities (13), (10), equality (12), and the fact that $T(r, u)$ is an increasing function of r, that

$$\rho = \overline{\lim_{i\to\infty}} \frac{\log T(r_i, u)}{\log r_i} \leq \overline{\lim_{i\to\infty}} \frac{\log T(|\lambda|^{m_i} r_0, u)}{\log r_i} \tag{14}$$

$$\leq \overline{\lim_{i\to\infty}} \frac{m_i \log n(1+\varepsilon) + \log T(r_0, u)}{\log r_i}$$

$$= \overline{\lim_{i\to\infty}} \frac{m_i \log n(1+\varepsilon)}{\log r_i}$$

$$= \frac{\log n(1+\varepsilon)}{\log |\lambda|} .$$

As ε can be chosen arbitrarily small, we have

$$\rho = u_\rho \leq \frac{\log n}{\log |\lambda|} ,$$

a finite number. Now we can apply Lemma 2 to the function $T(r, u)$ by choosing $\eta = \mu_u$ (the lower order of u). We have, for a sequence $\{r_m\}$, that

$$n(1+o(1)) = \frac{T(|\lambda| r_m, u)}{T(r_m, u)} \leq (1+o(1)) |\lambda|^{\mu_u} \tag{15}$$

for sufficiently large m.

By letting $m \to \infty$, we have

$$n \leq |\lambda|^{\mu_u} .$$

Hence

$$\mu_u \geq \frac{\log n}{\log |\lambda|} .$$

Our theorem follows from this and the fact that $\mu_u \leq \rho_u$.

4. Concluding Remark

In conclusion we mention that the argument used in the proof of Theorem 1 can be adopted to the study of the growth of the entire solutions for functional-differential of the form:

$$P(u(\lambda_1 + z), u(\lambda_2 + z), \cdots, u(\lambda_l + z)) = h(u^{(m)}(z)) \quad \text{with } \lambda_i (i = 1, 2 \cdots)$$

being arbitrary constants, and for this equation a conclusion similar to that of Theorem 1 can be drawn.

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