Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Global asymptotic stability of certain nonlinear autonomous differential equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **53** (1972), n.1-2, p. 39–45.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_53_1-2_39_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1972.

Equazioni differenziali non lineari. — *Global asymptotic stability of certain nonlinear autonomous differential equations.* Nota (*) di ROLF REISSIG, presentata dal Socio G. SANSONE.

RIASSUNTO. — In un recente lavoro S. Kasprzyk considera la globale asintotica stabilità di alcune equazioni differenziali non lineari del terzo ordine. I teoremi da lui provati, con l'aiuto del principio di Hartman–Olech possono essere dimostrati in forma più generale con una semplice estensione di alcuni risultati contenuti nel volume « Nichtlineare Differentialgleichungen höherer Ordnung ». Questi risultati possono essere derivati da un ben noto teorema fondamentale di stabilità con la costruzione di opportune funzioni di Liapunov.

In his paper [4] S. Kasprzyk proves the following stability theorems concerning the third order differential equations

$$x''' + ax'' + bx' + f(x) = 0 \qquad (a \text{ and } b \text{ positive constants}) \tag{1}$$

and

$$x''' + f(x'') + bx' + cx = 0 \qquad (b \text{ and } c \text{ positive constants}) \tag{2}$$

where the nonlinear term f is assumed to be continuously differentiable.

THEOREM 1. – *Let*

$$xf(x) > 0 \qquad \textit{for} \quad x \neq 0$$

$$\int_0^\infty \text{Min}\,[f(s)\,, -f(-s)]\,ds = +\infty$$

$$0 < f'(0) < ab$$

$$|f'(x)| \leqq ab \quad \textit{for each value } x\,.$$

Then the zero solution $x(t) \equiv 0$ of equation (1) is globally asymptotically stable.

THEOREM 2. – *Let*

$$uf(u) > 0 \qquad \textit{for} \quad u \neq 0$$

$$f'(0) > \frac{c}{b}$$

$$f'(u) \geqq \frac{c}{b} \quad \textit{for each value } u.$$

Then the zero solution $x(t) \equiv 0$ of equation (2) is globally asymptotically stable.

(*) Pervenuta all'Accademia il 18 agosto 1972.

The proof is based on a general result of Hartman–Olech [3] (cp. [7], Theorem 1.7/1.8); the essential idea consists in finding an adequate system of three first order equations which is equivalent to the basic equation (1) or (2). Stability theorems which are similar to Theorem 1 or Theorem 2 are contained in the book "Nichtlineare Differentialgleichungen höherer Ordnung" [7] (Theorem 4.7/5.7). They were derived with the aid of simple Liapunov functions according to the fundamental theorems of Liapunov's second method. It can be shown easily that the demonstration of these theorems carried out in [7] is compatible with a slight modification of their assumptions. By this means a generalization of the above Theorems 1 and 2 is attained.

The original text of the indicated theorems from [7] is as follows.

THEOREM 1′ (cp. [7], Theorem 4.7). – *The trivial solution of equation* (1) *is asymptotically stable in the whole if*

a) $xf(x) > 0$ *for* $x \neq 0$;

b) $f'(x) \leqq ab$ *(but* $f'(x) \not\equiv ab$ *on each x–interval).*

Now, condition b) is replaced by the more general one

b′) $f'(x) \leqq ab$, but $f'(x) < ab$ at least for two values x with opposite signs and with amounts smaller than ε (arbitrarily chosen).

THEOREM 2′ (cp. [7], Theorem 5.7). – *The trivial solution of equation* (2) *is asymptotically stable in the whole if*

a) $uf(u) > 0$ *for* $u \neq 0$;

b) $f'(u) > \frac{c}{b}$ *for each* u.

Now, condition b) is replaced by

b′) $f'(u) \geqq \frac{c}{b}$, but $f'(u) > \frac{c}{b}$ at least for two values u with opposite signs and with amounts smaller than ε (arbitrarily chosen).

The proofs given in [7] take pattern from a well-known stability theorem for autonomous vector differential equations

$$\mathbf{x}' = \mathbf{g}(\mathbf{x}) \quad [\mathbf{g}(0) = 0]: \tag{3}$$

THEOREM 3 (cp. [7], Theorem 1.11). – *The equilibrium position* $\mathbf{x}(t) \equiv 0$ *is globally asymptotically stable if all solutions are bounded for* $t \geqq 0$ *and if there exists a positive-definite Liapunov function* $V(\mathbf{x})$ *such that*

a) $V' = (\text{grad } V, \mathbf{g}(\mathbf{x})) \leqq 0$
and

b) *differential equation* (3) *with additional condition* $V' = 0$ *admits the only solution* $\mathbf{x}(t) \equiv 0$.

For example, the boundedness condition is fulfilled if

$$\lim_{|\mathbf{x}| \to \infty} V(\mathbf{x}) = +\infty .$$

The stability criterion of Theorem 3 is proper for applications; it seems to be more efficient than the criterion of Hartman-Olech which is applied in [4].

Proof of Theorem 1'. – The differential equation (1) is equivalent to system

$$x' = y \quad , \quad y' = -ay + z \quad , \quad z' = -f(x) - by \tag{4}$$

for which the following Liapunov function $V(x, y, z)$ was constructed by Ezeilo [1]:

$$2V = \frac{1}{b}[f(x) + by]^2 + \frac{2}{b}\int_0^x [ab - f'(s)]f(s)\,ds + z^2 .$$

V is positive-definite by virtue of condition a) and b') of Theorem 1'.

The total derivative $V' = \frac{dV}{dt}$ by means of system (4) is

$$\frac{dV}{dt} = -[ab - f'(x)]\,y^2 \leqq 0 .$$

With the aid of this Liapunov function the global asymptotic stability of system (4) can be proved via some special stability criterion of Pliss [5] (cp. [7], Theorem 1.6). This was done by Ezeilo [2] under conditions a) and b).

Now, assuming condition b') instead of b) we apply directly the above cited Theorem 3.

At first we investigate the boundedness of solutions. Since $V(x(t), y(t), z(t))$ is monotone-decreasing the terms

$$f(x(t)) + by(t) \quad \text{and} \quad z(t)$$

are bounded for $t \geqq 0$.

Let $x(t_0) > 0$ for some value $t_0 > 0$ and define

$$t_0' = \inf\{t \geqq 0 \mid x(\tau) > 0 \quad \text{for} \quad t < \tau \leqq t_0\}.$$

Integrating the third differential equation of system (4) we obtain

$$z(t_0) - z(t_0') = -\int_{t_0'}^{t_0} f(x(t))\,dt - b[x(t_0) - x(t_0')] .$$

Because of $|z(t)| \leqq Z\,(t \geqq 0)$ we conclude

$$x(t_0) < x(t_0') + \frac{2Z}{b} \leqq \mathrm{Max}(x_0, 0) + \frac{2Z}{b} \leqq X \quad , \quad x_0 = x(0) .$$

In an analogous way we should estimate $-x(t_0) > 0$.

An immediate consequence is

$$|y(t)| \leqq Y(t \geqq 0).$$

Subsequently, let us consider a solution of system (4) for which

$$[ab - f'(x(t))]\, y^2(t) \equiv 0 .$$

First case. - Let

$$f'(x(t_0)) < ab \qquad \text{for some} \quad t_0 \geq 0 .$$

Then, an interval $i = [t_0 , t_0'']$ must exist on which

$$y(t) \equiv 0 .$$

From equations (4) it follows successively that

$$y'(t) \equiv 0 \quad , \quad z(t) \equiv 0 \quad , \quad z'(t) \equiv 0 \quad , \quad f(x(t)) \equiv 0$$

$$(\text{i.e.} \quad x(t) \equiv 0) \quad \text{on this interval } i .$$

By virtue of the uniqueness theorem the considered solution is identical with the trivial one.

Second case. - Let

$$f'(x(t)) \equiv ab , \qquad \text{but not} \quad x(t) \equiv 0 \qquad (t \geqq 0).$$

Evidently,

$$\text{either} \quad x(t) < 0 \quad \text{or} \quad x(t) > 0 .$$

Assume that $x(t)$ is positive. Define $[x_0', x_0'']$ as the largest interval containing the initial point $x(0) = x_0$ where

$$f'(x) \equiv ab .$$

By virtue of condition b') x_0' must be a positive value. The component $x(t)$ of the considered solution of (4) belongs to this x-interval for $t \geqq 0$. That means: $x(t) \geqq x_0'$.

Furthermore, $f(x(t))$ can be represented as

$$f(x(t)) = f(x_0') + ab\,[x(t) - x_0'] .$$

Thus, differential equation (1) for $x = x(t)$ has the form

$$x''' + ax'' + bx' + abx = abx_0' - f(x_0') .$$

Consequently,

$$x(t) = x_0' - \frac{f(x_0')}{ab} + p \sin(\sqrt{b}\, t + \varphi) + q\, e^{-at}$$

$$(p > 0 \, , \; q \, , \; \varphi \leqq \pi \quad \text{constant values}) .$$

Let $\sqrt{b}\, t_n + \varphi = n\pi \quad (n = 1, 2, 3, \cdots)$; then

$$x(t_n) = x_0' - \frac{f(x_0')}{ab} + q e^{-at_n} < x_0' \quad \text{for sufficiently large } n,$$

contradictory to the above statement.

Therefore, the second case is impossible.

Proof of Theorem 2'. – Differential equation (2) is equivalent to the system

$$(5) \qquad u' = v - f(u) \quad , \quad v' = -bu - cw \quad , \quad w' = u$$

obtained by means of the transformation $u = x''$, $v = -cx - bx'$, $w = x'$ (cp. Pliss [6]).

We consider the Liapunov function $\Phi(u, v, w)$,

$$2\Phi = b\left[\frac{c}{b}\, u - f(u) + v\right]^2 + (bu + cw)^2 + 2c\int_0^u \left[f'(s) - \frac{c}{b}\right] s\, ds$$

which is positive-definite by virtue of condition b').

The total derivative $\frac{d\Phi}{dt} = \Phi'$ by means of (5) is

$$\Phi' = -b\left[f'(u) - \frac{c}{b}\right][v - f(u)]^2 \leqq 0 .$$

Since $\Phi(u(t), v(t), w(t))$ is monotone-decreasing the terms

$$h(t) = \frac{c}{b}\, u(t) - f(u(t)) + v(t) \quad \text{and} \quad bu(t) + cw(t)$$

are bounded for $t \geqq 0$.

From

$$u' + \frac{c}{b}\, u = h(t) \quad \text{where} \quad |h(t)| \leqq H$$

we derive

$$u(t) = u_0 e^{-\frac{c}{b}t} + \int_0^t e^{-\frac{c}{b}(t-\tau)} h(\tau)\, d\tau,$$

$$|u(t)| \leqq |u_0| + \frac{b}{c} H\left(1 - e^{-\frac{c}{b}t}\right) \leqq U \quad (t \geqq 0).$$

It follows immediately that $v(t)$ and $w(t)$ are bounded, too:

$$|v(t)| \leqq V \quad , \quad |w(t)| \leqq W \quad (t \geqq 0).$$

Now, let $(u(t), v(t), w(t))$ be a solution of system (5) satisfying the additional condition $\Phi' = 0$.

First case. – Assume that

$$f'(u(t_0)) > \frac{c}{b} \quad \text{for some} \quad t_0 \geqq 0 .$$

Then there must be an interval $i = [t_0 , t_0'']$ on which

$$f(u(t)) - v(t) \equiv 0 , \qquad \text{i.e.} \quad u'(t) \equiv 0 .$$

From equations (5) we obtain successively the following identities on i:

$$f'(u(t))\, u'(t) - v'(t) \equiv 0 , \qquad \text{i.e.} \quad v'(t) \equiv 0$$

$$bu'(t) + cw'(t) \equiv 0 , \qquad \text{i.e.} \quad w'(t) = u(t) \equiv 0$$

and finally

$$v(t) \equiv 0 \quad , \quad w(t) \equiv 0 .$$

The considered solution must be the trivial one.

Second case. – Assume that

$$f'(u(t)) \equiv \frac{c}{b} , \qquad \text{but not} \quad u(t) \equiv 0 \qquad (t \geqq 0) .$$

Then it is evident that

$$\text{either} \quad u(t) < 0 \quad \text{or} \quad u(t) > 0 .$$

Let $u(t)$ be positive. Define $[u_0' , u_0'']$ as the largest interval containing the initial value $u_0 = u(0)$ where

$$f'(u) \equiv \frac{c}{b} \cdot$$

The component $u(t)$, $t \geqq 0$, of the considered solution belongs to this interval; therefore, $u(t) \geqq u_0'$ (> 0 by virtue of condition b')).

The nonlinear term f with $u = u(t)$ may be written in the form

$$f(u) = f(u_0') + \frac{c}{b}\,(u - u_0') .$$

Now, using differential equation (2) for

$$x(t) = -\frac{v(t)}{c} - \frac{bw(t)}{c} \quad [x''(t) = u(t)]$$

we have

$$x''' + ax'' + bx' + cx = au_0' - f(u_0') \quad \left(\text{with abbreviation} \quad a = \frac{c}{b}\right) \cdot$$

After differentiating twice we obtain

$$u''' + au'' + bu' + abu = 0 .$$

It follows that $u(t)$ admits the representation

$$u(t) = p \sin(\sqrt{b}\, t + \varphi) + q e^{-at} \qquad (p, q, \varphi \text{ as above})$$

from which we conclude:

$$u(t_n) = q e^{-at_n} \to 0 \qquad (n \to \infty).$$

This is a contradiction to $u(t_n) \geqq u_0'$ (positive). The second case is impossible.

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