Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Dorin Ciumasu

On analytic solutions of linear partial differential equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **53** (1972), n.1-2, p. 18–24.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_53_1-2_18_0>

Analisi matematica. — *On analytic solutions of linear partial differential equations.* Nota (*) di DORIN CIUMAȘU, presentata dal Socio B. SEGRE.

RIASSUNTO. — Si studiano (con particolare riguardo al caso di una sola equazione) sistemi di equazioni lineari analitici alle derivate parziali del tipo (16) colle condizioni iniziali (17). Per essi si ottengono – sotto opportune condizioni come la (11) – teoremi di esistenza e di unicità, riducendo il problema ad un'equazione funzionale della forma (9), la quale viene poi risolta col metodo del punto fisso nello spazio delle serie di potenze formali od in quello delle serie convergenti, il che fornisce la richiesta soluzione quale limite di una successione iterativa.

The class of partial differential equations known to have analytic solutions has been considerably expanded and the techniques of tackling such problems has been developed in the last years [2–8, 10]. Recently, I presented a functional method for treating these problems in [2, 3]. In the present paper I consider thoroughly this method, obtaining specific results for linear systems. The auxiliary notions used here can be found in [1, 9, 11].

1. Let $I \neq \emptyset$, Card (I) $= n$, and $\mathbf{I} = N^I$. If $m = (m_i : i \in I) \in \mathbf{I}$, we denote

$$|m| = \sum_{i \in I} m_i \quad , \qquad m! = \prod_{i \in I} m_i! .$$

If $J \subset I$, we denote

$$|m|_J = \sum_{i \in J} m_i .$$

Let A be a commutative ring, with element 1, of characteristic zero, and $A[[X_i : i \in I]]$ be the ring of formal power series in the indeterminates $(X_i : i \in I)$ over A.

If $J \subset I$, and $u = (u_m : m \in \mathbf{I}) \in A[[X_i : i \in I]]$ then $\omega_J(u) = \inf\{|m|_J , u_m \neq 0\}$, if $u \neq 0$, and $\omega_J(u) = +\infty$, if $u = 0$, is the order of the formal series u with respect to the indeterminates $(X_i : i \in J)$. When $J = \{i\}$, $i \in I$, or $J = I$, we shall use the notation ω_i and ω, respectively.

The strict order of a formal series u is defined by $\Omega(u) = r = (r_i : i \in I) \in \mathbf{I}$ if $\omega_i(u) = r_i$, $i \in I$.

The sets

$$[\mathfrak{M}(\omega_J)]^k = \{u ; \omega_J(u) \geq k\}, \qquad k \geq 1$$

(*) Pervenuta all'Accademia il 25 luglio 1972.

form a fundamental system of neighbourhoods for a topology $T(\omega_J)$ over $A[[X_i : i \in I]]$, which is called the topology of the order ω_J. This topology is metrisable with respect to the distance

$$d(u, v) = \exp[-\omega_J(u - v)], \tag{1}$$
$$u, v \in A[[X_i : i \in I]],$$

and the topological ring $A[[X_i : i \in I]]$ is complete [1, 10].

2. Let R_+^* be the set of real strictly positive numbers and $\Gamma = (R_+^*)^I$. If $\alpha = (\alpha_i : i \in I) \in \Gamma$, we shall denote $|\alpha| = \sum_{i \in I} \alpha_i$.

Let A be a nondiscrete complete valuation ring. We define the mapping

$$\Phi : A[[X_i : i \in I]] \to R[[X_0]]; \qquad 0 \notin I,$$

such that

$$u(X) = \sum_{m \in I} \frac{u_m}{m!} X^m \in A[[X_i : i \in I]]$$

has the image

$$\Phi(u(X)) = \sum_{k \in N} \frac{M_k}{k!} X_0^k \in R[[X_0]],$$

where

$$M_k = \sup_{|m|=k} |u_m|, \quad k \in N, \qquad X_0 = \sum_{i \in I} X_i,$$

(Gårding's formal majorant).

We shall call a convergence bound of the formal series $u(X)$, the convergence radius of the power series $\Phi(u(X))$ and we shall denote it by $\rho(u)$.

For any $\alpha \in \Gamma$ we shall denote

$$\|u\|_{|\alpha|} = \Phi(u(\alpha))$$

and

$$C_{|\alpha|} = \{u ; \|u\|_{|\alpha|} < +\infty\}, \qquad C = \bigcup_{\alpha \in \Gamma} C_{|\alpha|}$$

the ring of the convergent series.

Then $(C_{|\alpha|}, \|\cdot\|_{|\alpha|})$ is a complete topological ring.

Let B be an open subset of A^I. The mapping $u : B \to A$ is analytic on B, if for any $a \in B$, there exists an element $u_a \in C$ such that $u(x) = u_a(x)$ in some neighbourhood of the point a.

3. We shall deal with the case of a linear partial differential equation. A similar method can be used for systems of linear equations.

Let be the equation

$$D^r u(x) = \sum_{s \in S} f_s(x) \cdot D^s u(x) + g(x), \tag{2}$$

where

$$r \in \mathbf{I} \quad , \quad S \subset \mathbf{I} , \qquad D^r = \partial^{|r|}/\partial x_1^{r_1} \cdots \partial x_n^{r_n} ,$$

and the mappings f_s $(s \in S)$ and g are analytical in some neighbourhood of the origin of A^I $(f_s , g : A^I \to A)$.

We seek a solution of the equation (2), satisfying the initial conditions

$$D_i^k u(x)\big|_{x_i=0} = 0 , \qquad 0 \leq k < r_i \quad , \quad i \in I , \tag{3}$$

where

$$D_i = \partial/\partial x_i , \qquad i \in I .$$

In order to prove the existence and uniqueness of the solution for the problem (2), (3) in the class of analytical mappings, we shall first determine the conditions for existence of a formal solution in $A[[X_i : i \in I]]$. This solution will appear as a limit in $T(\omega)$ of an iterative sequence $\{u^{(q)}(X) : q \in N\}$. Since there exists an element $\alpha \in \Gamma$ such that $u^{(q)}(X) \in C_{|\alpha|}$, it follows that this sequence is convergent in $C_{|\alpha|}$, and its limit satisfies (2), (3) and defines an analytical mapping in some neighbourhood of the origin of A^I.

4. Since the mappings f_s $(s \in S)$ and g are analytical in some neighbourhood of the origin, we have first to find a formal series $u(X) \in A[[X_i : \in I]]$, such that

$$D^r u(X) = \sum_{s \in S} f_s(X) \cdot D^s u(X) + g(X) , \tag{4}$$

$$\Omega(u(X)) \geq r , \tag{5}$$

where $f_s(X) , g(X) \in A[[X_i : i \in I]]$, $s \in S$, and D^m $(m \in \mathbf{I})$ denote the algebraic derivative in $A[[X_i : i \in I]]$.

We remark that unlike the nonlinear case, such a problem has always a sense in $T(\omega)$.

Consider the formal series

$$\varphi(u(X)) = \sum_{s \in S} f_s(X) \cdot D^s u(X) = \sum_{m \in \mathbf{I}} \varphi_m X^m . \tag{6}$$

It is obvious that the mapping

$$\varphi : A[[X_i : i \in I]] \to A[[X_i : i \in I]]$$

is linear and continuous in $T(\omega)$.

Define the mapping

$$D^{-r} : A[[X_i : i \in I]] \to A[[X_i : i \in I]]$$

such that

$$v(X) = \sum_{m \in \mathbf{I}} v_m X^m$$

has the image

$$\text{(7)} \qquad D^{-r} v(X) = \sum_{m \in I} \frac{m!}{(m+r)!} v_m X^{m+r}$$

(formal integration).

This mapping is linear, and since

$$\omega(uv) = \omega(u) + \omega(v) \quad ; \quad u, v \in A[[X_i : i \in I]],$$

it follows that

$$\text{(8)} \qquad \omega(D^{-r} v) = |r| + \omega(v).$$

Hence this mapping is continuous in $T(\omega)$.

It can be seen that the problem (4), (5) is equivalent to the functional equation

$$\text{(9)} \qquad u = D^{-r} \varphi(u) + \bar{g} \quad , \quad \bar{g} = D^{-r} g.$$

In view of (8), (6) and because

$$\omega\left(\sum_k u_k\right) \geq \inf_k \omega(u_k) \quad , \quad \omega(D^s u) \geq \omega(u) - |s|,$$

$$u_k, u \in A[[X : i \in I]],$$

we obtain

$$\text{(10)} \qquad \omega[D^{-r} \varphi(u)] \geq |r| + \inf_{s \in S} \{\omega(f_s) - |s|\} + \omega(u).$$

Since $A[[X_i : i \in I]]$ is a metric space with the distance (1) and the mappings φ, D^{-r} are linear, it follows:

$$d(D^{-r} \varphi(u), D^{-r} \varphi(v)) \leq L \cdot d(u, v),$$

where

$$L = \exp[-(|r| + \inf_{s \in S}(\omega(f_s) - |s|))].$$

Therefore the mapping $D^{-r}\varphi$ is a contraction if

$$\text{(11)} \qquad |r| + \omega(f_s) > |s|, \qquad s \in S.$$

Hence

THEOREM 1. *If the condition* (11) *is fulfilled, the problem* (4), (5) *has a unique solution* $u(X) \in A[[X_i : i \in I]]$. *This solution is a limit in* $T(\omega)$ *of any iterative sequence* $\{u^{(q)}(X) : q \in N\}$, *defined by*

$$\text{(12)} \qquad u^{(q+1)}(X) = D^{-r} \varphi(u^{(q)}(X)) + \bar{g}(X).$$

5. Some remarks about the previous theorem

1. If we denote

$$M = |r| + \inf_{s \in S} \{\omega(f_s) - |s|\},$$

then from (9), it follows

$$\omega(u - u^{(q)}) \geq q \cdot M + \omega(u^{(1)} - u^{(0)}).$$

Therefore $u^{(q)}(X)$ contains the polynomial part of degree not exceeding $qM + \omega(u^{(1)} - u^{(0)})$ of the exact solution. Moreover this solution can be expressed by the Neumann series

$$u(X) = \bar{g}(X) + (D^{-r}\varphi)\bar{g}(X) + (D^{-r}\varphi)^2 \bar{g}(X) + \cdots$$

which is a convergent series in $T(\omega)$.

2. When $r \in S$ and $\omega(f_r) = 0$, then it is necessary to require $1 - f_r(0)$, to have an inverse in A.

3. If we consider $A[[X_i : i \in I]]$ with the topology $T(\omega_J)$, $J \subset I$, then (11), becomes

$$|r|_J + \omega_J(f_s) > |s|_J, \qquad s \in S,$$

for some $J \subset I$.

4. In the case of linear systems of the form

$$D^{r(j)} u_j(X) = \sum_{k \in J} \sum_{s(j,k) \in S(j,k)} f_{s(j,k)}(X) \cdot D^{s(j,k)} u_k(X) + g_j(X), \qquad j \in J, \tag{13}$$

$$\Omega(u_j(X)) \geq r(j), \qquad j \in J, \tag{14}$$

where $r(j) \in \mathbf{I}$, $S(j,k) \subset \mathbf{I}$, we can use a similar method, and the relations (11) become

$$|r(j)| + \omega(f_{s(k,j)}(X)) > |s(k,j)|, \tag{15}$$

$$j, k \in J \quad , \quad s(k,j) \in S(k,j).$$

We note that the results hold if the set J is infinite, but in each of the equations (13) will appear only a finite number of series $u_k(X)$, $k \in J$.

6. Let us now consider the problem (4), (5) in C.

Since the mappings $f_s (s \in S)$ and g are analytical in some neighbourhood of the origin of A^I it follows that there exists $\delta \in R_+^*$ such that $\rho(f_s) \geq \delta$, $s \in S$ $\rho(g) \geq \delta$.

Let be $u^{(0)}(X)$ with $\rho(u^{(0)}(X)) \geq \delta$. From (12) we have $\rho(u^{(1)}(X)) \geq \delta$, and generally $\rho(u^{(q)}(X)) \geq \delta$. Choosing an element $\alpha \in \Gamma$, $|\alpha| < \delta$ it results

$$u^{(q)}(X) \in C_{|\alpha|}, \qquad q \in N.$$

Since this sequence is convergent in T (ω) (Theorem 1) it will be also convergent in $C_{|\alpha|}$, its limit being a convergent series which defines an analytical mapping in some neighbourhood of origin of A^I.

Hence,

THEOREM 2. *If* $f_s\,(s \in S)$ *and* g *are analytical mappings in some neighbourhood of the origin of* A^I, *and the conditions* (11) *are fulfilled, then there exists a unique analytical solution of the problem* (2), (3).

This solution appears as the limit in C of any iterative sequence defined by (12).

The remarks of § 5 are valid again. In particular when $\omega(f_s) = 0$, $s \in S$, we obtain well-known theorems ([8], Chap V).

This theorem also holds for systems

$$\text{(16)} \qquad D^{r(j)} u_j(x) = \sum_{k \in J} \sum_{s(j,k) \in S(j,k)} f_{s(j,k)}(x) \cdot D^{s(j,k)} u_k(x) + g_j(x), \qquad j \in J$$

$$\text{(17)} \qquad D_i^k u_j(x)\Big|_{x_i=0} = 0, \qquad 0 \le k < r_i(j) \quad , \quad i \in I \quad , \quad j \in J,$$

if $f_{s(j,k)}$ and g are analytical mappings and the condition (15) is satisfied.

We note that the Theorem 2 can also be formulated as follows:

THEOREM 3. *Let be the equation*

$$\text{(18)} \qquad \sum_{s \in S} f_s(x) \cdot D^s u(x) = g(x),$$

where $f_s\,(s \in S)$ *and* g *are analytical mappings in some neighbourhood of a point* $a \in A^I$. *If there exists an* $r \in S$ *such that* $f_r(a)$ *has an inverse in* A, *and*

$$|r| + \omega(f_s) > |s|, \qquad s \in S - \{r\},$$

then there exists a unique mapping $u(x)$, *analytic in the neighbourhood of the point* a, *which satisfies* (18) *and the conditions*

$$\text{(19)} \qquad D_i^k (u(x) - h(x))\Big|_{x_i=a_i} = 0, \qquad 0 \le k < r_i \quad , \quad i \in I,$$

where $h(x)$ *is a given mapping which is analytic in a neighbourhood of the point* a.

A similar result can be given for systems of equations.

I observe that for the equation

$$\frac{\partial^p u}{\partial t^p} = f(x, t) \frac{\partial^q u}{\partial t^q} + g(x, t),$$

$$\frac{\partial^k u}{\partial t^k}\Big|_{t=0} = h_k(x), \qquad 0 \le k < p,$$

$p < q$, f, g, h_k being analytical mappings, it is known that there exists a unique solution of Gevrey class [4, 10]. But as an immediate corollary of the Theorem 1, it results that there exists an analytic solution if the condition

$$p + \omega(f) > q$$

is fulfilled.

Finally we note that the generality of the method we have used, allows its use it for linear systems of partial differential equations in Banach spaces.

References

[1] BOURBAKI, *Algèbre*, Chap. 4, Hermann, Paris 1959.

[2] CIUMAŞU D., *Sur les équations aux dérivées partielles analytiques*, « Ann. St. Univ. Al. I Cuza Iaşi », *17*, 359–366 (1971).

[3] CIUMAŞU D., *Le problème généralisé de Cauchy dans un anneau de séries formelles et dans une algèbre de séries convergentes*, « Rev. Roum. De Math. pure et appl. » (to appear).

[4] FRIEDMAN A., *A new proof and generalizations of the Cauchy–Kowalewsky theorem*. « Trans. Amer. Math. Soc. », *98*, 1–20 (1961).

[5] GÅRDING L., *Une variante de la methode de majoration de Cauchy*, « Acta Math. », *114*, 143–158 (1965).

[6] HAIMOVICI A., *Sur la résolution de certaines systèmes d'équations aux dérivées partielles qui généralisent les systèmes de Cauchy–Kowalewsky*, « Publ. Math. », *9*, 198–212 (1962).

[7] HAIMOVICI A., *Sur un système qui généralise le système de Cauchy–Kowalewsky*, « Ann. St. Univ. Al. I. Cuza Iaşi », 33–56 (1962).

[8] HÖRMANDER L., *Linear partial differential operators*, Springer–Verlag, 1963.

[9] JURCHESCU M., *On the canonical topology on an analytic algebra and of an analytic module*. « Bull. Soc. Math. France », *93*, 129–153 (1965).

[10] PERSSON J., *New proofs and generalizations of two theorems by Lednev for Goursat problems*, « Math. Ann. », *178*, 184–20 (1968).

[11] ZARISKI O. and SAMUEL P., *Commutative algebra*. II Princeton, 1960.