Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Jerald P. Dauer

On the Optimization of Linear Control Systems Using the Contraction Principle

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **53** (1972), n.1-2, p. 100–103.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_53_1-2_100_0>

Teoria dei controlli. — *On the Optimization of Linear Control Systems Using the Contraction Principle.* Nota (*) di JERALD P. DAUER, presentata dal Socio G. SANSONE.

RIASSUNTO. — Il problema di ottimizzare un sistema lineare di controllo con un funzionale quadratico di costo è risolto usando il principio di contrazione. Per trovare la soluzione della corrispondente espressione ottimizzante è usato un metodo iterativo e sono dati limitazioni per l'errore relativo alle iterate.

1. INTRODUCTION

The problem of optimizing a linear control system with quadratic cost functional has been approached in the context of functional analysis by several authors (see [1–3]). Freeman [1] used this approach to show that the optimum control must satisfy an integral equation. He then developed conditions for solving this equation using the contraction mapping principle and thereby produced an iterative computing algorithm for obtaining the optimum control.

The purpose of this paper is to improve Freeman's contraction principle approach for a more restrictive class of control systems. By using a renorming technique, we are able to show general convergence of Freeman's algorithm for these systems. This approach eliminates the size requirement on the integral operator and instead makes use of its linearity. Both approaches give bounds on the normdistance between the optimum control (or cost functional) and its n^{th} iterate. As was pointed out by Freeman, the type of system we consider occurs in many practical situations. For the reader's convenience we shall retain Freeman's notation.

2. FORMULATION OF THE PROBLEM

We consider the linear control system

$$\begin{aligned} \dot{x}(t) &= \mathrm{A}(t)\,x(t) + \mathrm{B}(t)\,u(t) \\ c(t) &= \mathrm{C}(t)\,x(t). \end{aligned} \tag{1}$$

The state vector $x(t)$ is n dimensional, the control vector $u(t)$ is m dimensional and the output vector $c(t)$ is r dimensional. The piecewise continuous matrix functions A, B, C have consistent dimensions. The object is to control system (1) so that the output $c(t)$ is close to a specified piecewise continuous output $c_d(t)$. The measure of error $e(t)$ is taken to be

$$e(t) = c_d(t) - c(t).$$

(*) Pervenuta all'Accademia il 3 luglio 1972.

In striving for this objective, we wish to minimize the quadratic cost functional

$$J(u) = \int_{t_0}^{T} [e'(t) Q(t) e(t) + u'(t) R(t) u(t)] \, dt .$$

We assume Q and R are piecewise continuous and let a' denote the transpose of a. It is further assumed that $Q(t)$ is positive semidefinite and $R(t)$ is positive definite. As is consistent with this problem, we consider control functions u such that $\int_{t_0}^{T} |u(t)|^2 \, dt < \infty$.

It was shown by Freeman [1, Eq. (21)] that the optimum control is a solution of the integral equation

$$u = R^{-1} L^* Qz - R^{-1} L^* QLu , \tag{2}$$

called the *optimizing equation*. We now specify the terms involved in this equation. Let $\Phi(t, t_0)$ denote the fundamental matrix solution of $x' = A(t) x$ such that $\Phi(t_0, t_0)$ is the identity matrix. The function z is defined by

$$z(t) = c_d(t) - C(t) \Phi(t, t_0) x_0 .$$

The operator L is defined by

$$Lv = \int_{t_0}^{t} C(t) \Phi(t, s) B(s) v(s) \, ds$$

and its adjoint L^* is given by

$$L^* v = \int_{t}^{T} [C(t) \Phi(t, s) B(s)]' v(s) \, ds .$$

The bounded, piecewise continuous inverse of R is denoted by R^{-1}.

3. SOLVING THE OPTIMIZING EQUATION

In order to solve equation (2) we obtain a sequence of successive approximations

$$\begin{aligned} u_0 &= 0 \\ u_1 &= R^{-1} L^* Qz \\ u_2 &= R^{-1} L^* Qz - R^{-1} L^* QLu_1 \\ &\cdots\cdots\cdots\cdots \\ u_{n+1} &= R^{-1} L^* Qz - R^{-1} L^* QLu_n . \end{aligned} \tag{3}$$

Since u_0 and z are piecewise continuous we have that the functions $u_1, u_2, \cdots, u_n, \cdots$ are all piecewise continuous. Further, the operator $R^{-1} L^* Q$ is bounded, say by N, with respect to the operator norm corresponding to the norm defined by

$$\|v\| = \sup_{t_0 \le t \le T} |v(t)| . \tag{4}$$

Let

$$M = 2 N \|C\| \|\Phi\| \|B\|$$

and define

$$\|v\|_M = \sup_{t_0 \le t \le T} e^{-M(t-t_0)} |v(t)| . \tag{5}$$

Then for all $t_0 \le \tau \le T$ we have

$$\begin{aligned} & |e^{-M(\tau-t_0)} R^{-1} L^* QL(u(\tau) - v(\tau))| \\ & \le N \sup_{t_0 \le t \le T} |e^{-M(t-t_0)} L(u(t) - v(t))| \\ & = N \sup_{t_0 \le t \le T} \left| \int_{t_0}^{t} e^{-M(t-t_0)} C(t) \Phi(t, s) B(s) (u(s) - v(s)) \, ds \right| \\ & \le \frac{M}{2} \sup_{t_0 \le t \le T} \int_{t_0}^{t} e^{-M(t-s)} \, ds \cdot \|u - v\|_M \\ & \le \frac{1}{2} \|u - v\|_M . \end{aligned}$$

Hence

$$\|R^{-1} L^* QL(u - v)\|_M \le \frac{1}{2} \|u - v\|_M . \tag{6}$$

Proceeding as in the proof of the contraction mapping principle we obtain

$$\|u_{n+1} - u_n\|_M \le \frac{1}{2} \|u_n - u_{n-1}\|_M \le \cdots \le \frac{1}{2^n} \|u_1 - u_0\|_M .$$

Hence for any $m > n$ we have

$$\begin{aligned} \|u_m - u_n\|_M & \le \|u_m - u_{m-1}\|_M + \cdots + \|u_{n+1} - u_n\|_M \\ & \le \frac{1}{2^n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right) \|u_1 - u_0\|_M \\ & \le \frac{1}{2^{n+1}} \|u_1 - u_0\|_M . \end{aligned}$$

Therefore, the sequence of successive approximations $\{u_n\}$ is Cauchy with respect to $\| \;\|_M$. Since this norm is equivalent to the norm in (4) we have that the sequence $\{u_n\}$ is uniformly convergent. Let the measurable function u^* be defined by

$$u^*(t) = \lim_{n \to \infty} u_n(t) . \tag{7}$$

Dominated convergence shows that

$$\int_{t_0}^{T} |u^*(t)|^2 \, dt < \infty .$$

To see that u^* is a solution of (2) we note that

$$\begin{aligned} \|u^* - (R^{-1} L^* Qz - R^{-1} L^* QLu^*)\|_M &\leq \|u^* - u_{n+1}\|_M + \|u_{n+1} - (R^{-1} L^* Qz - R^{-1} L^* QLu^*)\|_M \\ &\leq \|u^* - u_{n+1}\|_M + \|R^{-1} L^* QL(u_n - u^*)\|_M \\ &\leq \|u^* - u_{n+1}\|_M + \frac{1}{2}\|u_n - u^*\|_M \end{aligned}$$

and use the uniform convergence of $\{u_n\}$ to u^*. Equation (6) shows that u^* is the unique solution of (2). Hence we have proved the following result.

PROPOSITION. *The function u^* is the unique solution of the optimizing equation.*

Remark. We can now proceed in the same manner as Freeman [1, § 3.1] to obtain bounds on the accuracy of the n^{th} iterate. However, we are able to use the convenient sup norm (4) and obtain the following bounds on the iterates of the optimum control.

$$\begin{aligned} \|u^* - u_n\| &\leq e^{M(T-t_0)} \|u^* - u_n\|_M \\ &\leq e^{M(T-t_0)} \left[\left(\frac{1}{2}\right)^n \Big/ \left(1 - \frac{1}{2}\right)\right] \|u_1 - u_0\|_M \\ &\leq e^{M(T-t_0)} \left(\frac{1}{2}\right)^{n-1} N \|z\|_M \\ &\leq e^{M(T-t_0)} \left(\frac{1}{2}\right)^{n-1} N \|z\| . \end{aligned}$$

The distance of the approximate cost functional $J(u_n)$ from its optimum value $J(u^*)$ is bounded similarly [1, Eq. (33)]

$$|J(u^*) - J(u_n)| \leq N^2 \|R\| \left(\frac{1}{2}\right)^{n-2} .$$

REFERENCES

[1] FREEMAN E. A., *On the Optimization of Linear, Time-Variant, Multivariable Control Systems Using the Contraction Mapping Principle*, « Journal of Optimization Theory and Applications », *3* (6) (1969).

[2] BALAKRISHNAN A. V., *An Operator Theoretic Formulation of a Class of Control Problems and a Steepest-Descent Method of Solution*, « SIAM Journal on Applied Mathematics », *1* (2) (1963).

[3] HSIEH H. C., *Synthesis of Adaptive Control Systems by Function Space Methods*, « Advances in Control Systems, Theory and Applications », *2*, Edited by C. T. Leondes, Academic Press, New York 1965.