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M. Margaret Eich, Stanley E. Payne

Nonisomorphic Symmetric Block Designs Derived from Generalized Quadrangles

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Geometrie finite. — *Nonisomorphic Symmetric Block Designs Derived from Generalized Quadrangles.* Nota di M. MARGARET EICH, e STANLEY E. PAYNE, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si mostra che disegni simmetrici derivati da quadrangoli generalizzati non isomorfi fra loro sono essi pure non isomorfi, almeno nel caso di quadrangoli aventi in comune certe proprietà astratte con quelli noti. Inoltre, con l'eccezione di una particolare classe di quadrangoli, ogni isomorfismo fra un siffatto disegno ed uno di tipo noto è indotto da un isomorfismo fra i relativi quadrangoli.

1. INTRODUCTION

Theorem 4.1 of Payne [7] says that in certain special cases symmetric block designs derived from nonisomorphic generalized quadrangles must themselves be nonismorphic. The purpose of this note is to elaborate upon that thema, with our main result being that the result just stated is true for all known quadrangles.

We refer the reader to [4], [5], [6] and [7] for definitions and results pertaining to generalized quadrangles, but we now review in a unified fashion the derivation of block designs from quadrangles. Let P_4 be a generalized quadrangle of order (s, t). Let A be an incidence matrix of P_4 whose columns are indexed by the points of P_4 and whose rows are indexed by the lines of P_4. It follows from calculations in [4] that $\mathrm{M} = \mathrm{A}'\mathrm{A}$ is a matrix of size $v = (1 + s)(1 + st)$ satisfying $\mathrm{M}[\mathrm{M} - (s + t)\mathrm{I}] = (1 + t)\mathrm{J}$, where J is the matrix of 1's. For an arbitrary constant c, let $\mathrm{D} = \mathrm{M} - c\mathrm{I}$. Then $\mathrm{D}^2 = \mathrm{D}'\mathrm{D}$ is a linear combination of I and J precisely when $c = \frac{1}{2}(s + t)$. From now on let $c = (s + t)/2$. Then

$$\mathrm{D}^2 = \mathrm{D}'\mathrm{D} = c^2\,\mathrm{I} + (1 + t)\,\mathrm{J}\,. \tag{1}$$

The diagonal of M is $(1 + t)\,\mathrm{I}$, hence D is a nonnegative integral matrix provided $s \leq t + 2$ and $s \equiv t \pmod 2$. Moreover D is the incidence matrix of a (v, k, λ)–design precisely when $s = t$ or $s = t + 2$, in which cases $v = (1 + s)(1 + st)$, $k = 1 + t + c^2$, and $\lambda = 1 + t$ (c.f. [10]).

For each prime power q there are known (c.f. [5] and [7]) generalized quadrangles of order (s, t) where

i) $s = t = q$

and ii) $s = q + 1$, $t = q - 1$.

In Section 2 a simple proof is given that any isomorphism between (v, k, λ)–designs derived from quadrangles of order $(q + 1, q - 1)$ (where

(*) Nella seduta del 13 maggio 1972.

q is not necessarily a prime power) must be induced by an isomorphism between the underlying quadrangles. In addition, some general results are given which are applied in Section 3 to the known examples with $s = t$. Here it is no longer true that any isomorphism of a derived design must be induced by one of the underlying quadrangle, but the exceptions noted in [7] remain the only known ones.

Finally, in Section 4 a brief review is given of some of the known quadrangles of order s with an indication of which ones are nonisomorphic.

2. General Results

Let P_i be a generalized quadrangle of order $(q+1, q-1)$, and let B_i be the (v, k, λ)–design derived from P_i, $i = 1, 2$. So the points and blocks of B_i are indexed by the points $x_1^{(i)}, \cdots, x_v^{(i)}$ of P_i, with the block indexed by $x_j^{(i)}$ being incident with the point indexed by $x_k^{(i)}$ if and only if $x_j^{(i)} \neq x_k^{(i)}$ and $x_j^{(i)}$ and $x_k^{(i)}$ are collinear in P_i, $i = 1, 2$.

THEOREM 2.1. *Any isomorphism τ from B_1 to B_2 is induced by a unique isomorphism from P_1 to P_2, at least if $s \geq 3$.*

Proof: Let τ be an isomorphism from B_1 to B_2. Then τ is a pair (f, g), where f is a bijection from the points of B_1 to the points of B_2, g is a bijection from the lines of B_1 to the lines of B_2. Hence f and g are really bijections from the points of P_1 to the points of P_2 which satisfy

(2) $x \sim y$ if and only if $f(x) \sim g(y)$, where "$\sim$" means "is collinear with but not equal to" in P_1 or P_2, whichever is appropriate.

We claim that $f = g$ and f preserves collinearity, i.e. f determines a unique isomorphism from P_1 to P_2. So first suppose that f does not preserve collinearity. Then there must be distinct collinear points x and y of P_1 such that $f(x)$ and $f(y)$ are not collinear in P_2. Let $z_1, \cdots, z_q$ be the remaining points on the line of P_1 through x and y. Then $g(z_1), \cdots, g(z_q)$ must be precisely the points of P_2 collinear with both $f(x)$ and $f(y)$. Since $x \sim y$, clearly $g(x) \sim f(y)$. So $g(x)$ lies on a line through $f(y)$, say the one through $g(z_1)$. But $f(z_i) \sim g(x)$ and $f(z_i) \sim g(z_1)$, for $i = 2, \cdots, q$. Hence $f(y), g(x), g(z_1), f(z_2), \cdots, f(z_q)$ must be the distinct points of a line L. For $2 \leq i, j \leq q$, with $i \neq j$, it must be that $g(z_i) \sim f(z_j)$, $g(z_i) \sim f(y)$, so that $g(z_i)$ is on L (since $q \geq 3$). But then $f(x) \sim g(z_i)$ for $1 \leq i \leq q$ implies $f(x)$ is on L, so $f(x) \sim f(y)$, a contradiction. Hence f must preserve collinearity. With no triangles in P_2 it follows easily from (2) that $f = g$.

For the remainder of this section let P_4 denote a generalized quadrangle of order s. If x and y are distinct points of P_4, there are $1 + s$ points $z_0, \cdots, z_s$ collinear with both of them. The *trace* of x and y is defined by $tr\{x, y\} = \{z_0, z_1, \cdots, z_s\}$. As in [7] we say that the pair (x, y) is *regular* provided each point collinear with at least two of the points in $tr\{x, y\}$ is actually collinear with all of them. And a point x is regular provided (x, y) is regular for all points $y \neq x$. The following theorem, which essentially is a corollary of the proof of Lemma 2.2 of [7], is Theorem 3.8 of [8].

THEOREM 2.2. *Suppose that x is a regular point of P_4. If every point collinear with x is regular, then every point of P_4 is regular. The dual for lines is also valid.*

Each of the known quadrangles of even order (or its dual) has a special point x_∞ which is regular, and such that each of the lines $L_0, \cdots, L_s$ through x_∞ is also regular. (Conversely, the existence of such a point in a quadrangle of order s implies s is even, but the proof we have does not readily fit into the context of this note). Furthermore, the collineation group G of P_4 is transitive on the points different from x_∞ lying on L_i, for each $i = 0, 1, \cdots, s$; and G is transitive on the points not collinear with x_∞. Such a point x_∞ of P_4 will be called a *pivotal* point of P_4.

THEOREM 2.3. *Let P_4 have a pivotal point x_∞. Then either all points and lines of P_4 are regular and P_4 is of Type I in the notation of* [5], *and s is a power of* 2; *or the points not collinear with x_∞ are irregular, and* G *must fix x_∞.*

Proof: First suppose some point not collinear with x_∞ is regular. Then since collineations preserve regularity, all points not collinear with x_∞ are regular. Using Lemma 2.1 of [7] it is easy to show that all points of P_4 are regular. So by Benson [1] (also see Singleton [11]) P_4 is of Type I with s a power of 2 (since some line of P_4 is regular).

Hence we may suppose each point not collinear with x_∞ is irregular. If G moves x_∞ to some point $y \neq x_\infty$, then y must be pivotal, hence regular and on some line through x_∞, say L_i. Then since each line through y is regular, the transitivity of G on the points of L_i implies that all lines meeting L_i must be regular. By Theorem 2.2 all lines of P_4 are regular, and the dual of P_4 must be of Type I. Then if s were odd, all points would be irregular by Benson (c.f. [1] and [2]), contradicting the regularity of x_∞. If s were a power of 2, again by [1] all points would be regular, a contradiction. Hence G must fix x_∞.

If x and y are collinear points of P_4 (including $x = y$), we write $x \sim y$, otherwise $x \nsim y$. A point x_∞ of P_4 is called a *center of irregularity* provided the following is true: If y and z are distinct collinear points with $y \nsim x_\infty$ and $z \nsim x_\infty$, then there is some point w such that $w \sim z$ and (y, w) is an irregular pair.

The following Lemma is a key step in the proof of our main Theorem.

LEMMA 2.4. *Suppose P_4 has a center of irregularity. Let Q be a permutation of the points of P_4 satisfying the following*:

(i) *$y \sim y^Q$ for all points y of P_4*;

(ii) *$y \sim w$ if and only if $y^Q \sim w^{Q^{-1}}$, for all points y, w of P_4*;

(iii) *If (y, w) is any irregular pair of points, then $w \nsim y^{Q^{-1}}$.*

Then Q is the identity permutation.

Proof: Suppose x_∞ is a center of irregularity, and let y be a point such that $y \not\sim x_\infty$ and $y \neq y^{Q^{-1}}$, so $y^Q \neq y$. By (i) $y \sim y^{Q^{-1}}$. If $y^{Q^{-1}} \not\sim x_\infty$, there must be some point w such that $w \sim y^{Q^{-1}}$ and (y, w) is irregular. But this is impossible by (iii). Hence if $y \neq y^{Q^{-1}}$, $y \not\sim x_\infty$, then $y^{Q^{-1}} \sim x_\infty$. Indeed, $y^{Q^{-1}}$ must be the unique point on the line L through y and $y^{Q^{-1}}$ which is collinear with x_∞. Now if $y^Q \neq y^{Q^{-1}}$, then by (i) and (ii) y^Q must lie on L. Hence $y^Q \not\sim x_\infty$, so $(y^Q)^{Q^{-1}} = y$ must be collinear with x_∞. This is clearly impossible. Hence if $y \not\sim x_\infty$, then $y^Q = y^{Q^{-1}}$, i.e. Q^2 fixes each point not collinear with x_∞. Then from (ii) applied to x_∞ and each point not collinear with x_∞, it follows that Q^2 fixes x_∞. Then (ii) implies that $Q^2 = I$, i.e. Q is a collineation of order 2.

We now claim Q fixes x_∞. For suppose $x_\infty^Q = z \neq x_\infty$. Then $z \sim x_\infty$. Let L_∞ be the line through z and x_∞. Since $Q^2 = I$, $z^Q = x_\infty$ and Q must fix L_∞. Also z must be a center of irregularity. It now follows for z just as it did for x_∞ that if y is a point such that $y^Q \neq y$ and $y \not\sim z$, then y^Q is the unique point on the line L through y and y^Q which is collinear with z. Hence any point not collinear with either z or x_∞ must be fixed by Q. Since each line not meeting L_∞ contains at least two points not collinear with x_∞ or z (the unique quadrangle of order 2 has all its points and lines regular), it is clear that every line of P_4 not meeting L_∞ must be fixed by Q. It follows readily that $Q = I$, which contradicts the assumption that $x_\infty^Q \neq x_\infty$.

Finally, since Q fixes x_∞, it must map each point not collinear with x to another such point. This is clearly impossible unless $Q = I$.

COROLLARY 2.5. *If P_4 is a generalized quadrangle of order s in which each pair of noncollinear points is irregular, then any permutation Q of the points of P_4 satisfying* (i), (ii), *and* (iii) *of Lemma 2.4 must be the identity permutation.*

Proof: Each point of P_4 is a center of irregularity.

For $i = 1, 2$ let P_i be a generalized quadrangle of order s; let A_i be an incidence matrix of P_i with points labeling columns and lines labeling rows; and let $B_i = A_i' A_i - sI$ so that B_i is an incidence matrix of the (v, k, λ)-design (also denoted B_i) derived from P_i.

THEOREM 2.6. *If P_2 has a center of irregularity, then any isomorphism from B_1 to B_2 is induced by an isomorphism from P_1 to P_2.*

Proof: Suppose that P_2 has a center of irregularity and that B_1 and B_2 are isomorphic. Hence there are permutation matrices P and Q such that $PB_1Q = B_2$. By reordering the points of P_1 so that its new incidence matrix is PA_1P^{-1}, we may suppose $B_1 = B_2Q$ for some permutation Q. If $Q = I$, we are done.

So suppose $Q \neq I$. Since $B_1 = B_2Q$ is symmetric and contains I, Q is a permutation of the points of B_2, i.e. of the points of P_2, satisfying conditions (i) and (ii) of Lemma 2.4. Just as in the proof of Theorem 4.1 of [7], the fact that P_1 has no triangles translates directly into condition (iii) of Lemma 2.4. Hence by Lemma 2.4, $Q = I$.

3. APPLICATIONS TO THE KNOWN QUADRANGLES OF ORDER s

In spite of the effort made in [8] to provide a method for constructing new quadrangles of order s, the only ones known to us are those listed in [5] as Type I, II, or III, or the dual of one Type III. In [1] Benson showed that a block design derived from a quadrangle of Type I completely determines the quadrangle, but some collineations of the design are not induced by collineations of the quadrangle. Moreover, if s is an odd prime power, the dual of a Type I quadrangle (i.e. one of Type II) has all its points irregular. Hence Theorem 2.6 applies in this case. The remaining quadrangles are those of Type III which are not of Type I, and their duals.

Since we need to compute in a detailed way with these quadrangles, we review their construction in some detail, using the slightly revised coordinatization which appears in [8].

A *complete oval* of PG $(2, 2^e)$ is a set of $2 + 2^e$ points no three of which are collinear. Let Ω be a complete oval of PG $(2, 2^e)$, and let N be any point of Ω. Then the set $\Omega_N = \Omega \setminus \{N\}$ is an *ovoid* with nucleus N: through each point of Ω_N are s secants and a unique tangent, with all tangents meeting at N. By choosing coordinates suitably we may assume without loss of generality that a given Ω_N consists of the points with nonhomogeneous coordinates as follows:

$$\Omega_N = \{(1, c, c^\alpha) \mid c \in GF(2^e)\} \cup \{0, 0, 1)\}, \tag{3}$$

with nucleus $N = (0, 1, 0)$, where α is a permutation of the elements of $F = GF(2^e)$ such that $0^\alpha = 0$, $1^\alpha = 1$, and

$$\frac{c_0^\alpha - c_1^\alpha}{c_0 - c_1} \neq \frac{c_0^\alpha - c_2^\alpha}{c_0 - c_2}$$

for distinct $c_0, c_1, c_2 \in F$. As noted in [8], interchanging the point $X = (1, c, c^\alpha)$ of Ω_N with N yields an ovoid Ω_X which may be associated with the permutation

$$\alpha_c : x \rightarrow \frac{x[c^\alpha + (c + x^{-1})^\alpha]}{c^\alpha + (c + 1)^\alpha}. \tag{4}$$

Interchanging $X = (0, 0, 1)$ with N yields an ovoid which may be associated with α^{-1}.

With each permutation α of the type described above (i.e. with each ovoid Ω_N) there is associated a generalized quadrangle $P(\alpha)$ (to which we refer as a Tits quadrangle; c.f. [3], p. 304) described as follows. $P(\alpha)$ has points (x, y, z), (x, y), (x), and (∞) for $x, y, z \in F$, and lines $[u, v, w]$, $[u, v]$, $[u]$, and $[\infty]$ for $u, v, w \in F$. Incidence (which differs only slightly from that in [5] and agrees with that in [8]) is defined as follows: (∞) is on $[\infty]$ and on $[u]$; (x) is on $[\infty]$ and on $[x, y]$; (u, v) is on $[u]$ and on $[u, v, w]$; (x, y, z) is on $[x, y]$ and on $[u, z + u^\alpha x, y + ux]$; this is for all $x, y, z, u, v, w \in F$. Fig. 1 is helpful in picturing these incidences.

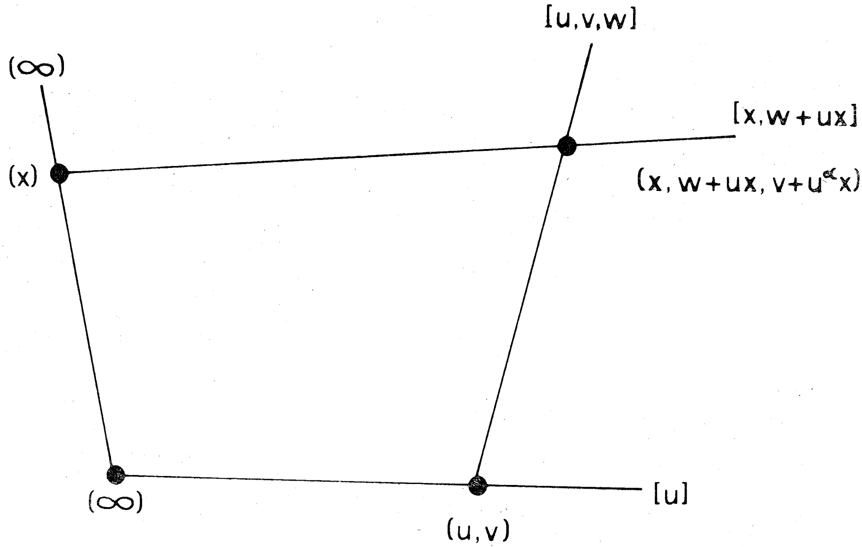

Fig. 1.

The quadrangle $P(\alpha)$ is of Type I precisely when $\alpha = 2$, i.e. $\alpha : x \to x^2$. So the purpose of this section is to prove the following:

THEOREM 3.1. *Let α be a permutation of* F *as described above but with $\alpha \neq 2$, and let $P(\alpha)$ be the associated Tits quadrangle. Then (∞) is a center of irregularity for $P(\alpha)$, and $[\infty]$ is a center of irregularity for the dual of $P(\alpha)$.*

Proof: We first note that our representation of $P(\alpha)$ is derived from that in [5] by making the following two changes: the line $[u, v, w]$ and the point (x, y) of [5] are now labeled $[u, w, v]$ and $(y^{\alpha^{-1}}, x)$, respectively, and incidence is translated directly to the new notation. For $P(\alpha)$ we must show the following: If X and Y are distinct collinear points such that neither one is collinear with (∞), then there exists a point W collinear with Y such that (X, W) is irregular. Since the group G_∞ of collineations of $P(\alpha)$ fixing (∞) is transitive on the points not collinear with (∞) (c.f. [6], Lemma 2), we may assume that $X = (0, 0, 0)$. For each $u \in F$, the line $[u, 0, 0]$ passes through $(0, 0, 0)$ and contains the points $(x, ux, u^\alpha x)$, $x \in F$, and $(u, 0)$. The other line through $(0, 0, 0)$ is $[0, 0]$, containing points $(0, 0, z)$, $z \in F$, and (0). Hence it is clear that the collineations given in Lemma 3 of [6] are transitive on those points not collinear with (∞) and different from $(0, 0, 0)$ which lie on a given line through $(0, 0, 0)$. Let $c \in \mathrm{F}$ be such that $c^\alpha \neq c^2$. From Lemma 2.4 of [7] we know that $\mathrm{Z} = (1, c, c)$ is such that (X, Z) is irregular. Furthermore, Z is collinear with some point on each of the lines through $(0, 0, 0)$. Except for the line $[c^{\alpha^{-1}}, 0, 0]$ containing $(c^{\alpha^{-1}}, 0)$, this point is not collinear with (∞). It follows that for each point $\mathrm{Y} \not\sim (\infty)$ lying on a line through $(0, 0, 0)$ but different from $[c^{\alpha^{-1}}, 0, 0]$, there is some collineation π of the type given in Lemma 3 of [6] for which $\mathrm{Z}^\pi \sim \mathrm{Y}$ and $(\mathrm{X}, \mathrm{Z}^\pi)$ is irregular. For the line $[c^{\alpha^{-1}}, 0, 0]$, we know by Lemma 2.4 of [7] that $\mathrm{Z}' = \left(\left(\frac{c^\alpha + c}{c+1}\right)^{\alpha^{-1}}, c^\alpha + c\right)$ is such that $(\mathrm{X}, \mathrm{Z}')$ is irregular, and $\mathrm{Z}' \not\sim (c^{\alpha^{-1}}, 0)$. Hence (∞) is a center of irregularity for $P(\alpha)$.

For the dual of $P(\alpha)$ we first observe that there must be at least three elements $c \in \mathrm{F}$ for which $c^\alpha \neq c^2$. The assumption that $\alpha \neq 2$ implies that $e \geq 3$, and clearly there are at least two distinct elements c_1, c_2 with $c_i^2 \neq c_i^\alpha$. Suppose that these are the only such elements, so that $c_1^\alpha = c_2^2$ and $c_2^\alpha = c_1^2$, and $u^\alpha = u^2$ for $u \neq c_1, c_2$. Then $\frac{c_1^\alpha + u^\alpha}{c_1 + u} \neq \frac{c_1^\alpha}{c_1}$ implies $u \neq \frac{c_2^2}{c_1}$. So $u = \frac{c_2^2}{c_1}$ is a third element of F with $u^\alpha \neq u^2$, contradicting our assumption. Hence there are at least three elements of the desired type.

In order to complete the proof of the Theorem we need to establish the following lemma, which has some independent interest.

LEMMA 3.2. *In $P(\alpha)$, $\alpha \neq 2$, two nonconcurrent lines form a regular pair if and only if some line through (∞) is in their trace.*

Proof: It is easy to check the fact (first noted in [8]) that each line through (∞) is regular. So by the dual of Lemma 2.1 of [7] the "if" part of the present Lemma is true. Now suppose that L_1 and L_2 are nonconcurrent lines with no line through (∞) in their trace. Using the collineations given in Lemmas 2 and 3 of [6], we may assume that L_1 and L_2 fall into one of the following cases:

Case 1. $\mathrm{L}_1 = [u, 0, 0]$, $\mathrm{L}_2 = [v, 1, 0]$, where $u \neq v$,

Case 2. $\mathrm{L}_1 = [0, 0]$, $\mathrm{L}_2 = [c, 0, 1]$.

We first consider case 1. Then $\mathrm{L}_3 = \left[0, \frac{u+v}{u^\alpha + v^\alpha}\right]$ is concurrent with $\left[v, 1, \frac{u+v}{u^\alpha + v^\alpha}\right]$ and with $\left[u, 0, \frac{u+v}{u^\alpha + v^\alpha}\right]$, both of which are in the trace of L_1 and L_2. We claim that there is some line $\mathrm{L} = [x, y, z]$ in the trace of L_1 and L_2 $\left(\text{with } u \neq x \neq v, \frac{z}{u+x} = \frac{y}{u^\alpha + x^\alpha}, \frac{z}{v+x} = \frac{y+1}{v^\alpha + x^\alpha}\right)$ for which L and L_3 are not concurrent. For if L were also concurrent with L_3, then $z = \frac{u+v}{u^\alpha + v^\alpha}$. Eliminating z and y from these three equations yields $0 = \mathrm{P}(x) = x^2 (u^\alpha + v^\alpha) + u^2 (x^\alpha + v^\alpha) + v^2 (x^\alpha + u^\alpha)$. Hence it suffices to show that there is some x such that $\mathrm{P}(x) \neq 0$. To do this it seems necessary to consider the various cases arising according as u and v are independently equal to zero, carried by α to their squares, or not carried by α to their squares.

In all cases there is an x such that $P(x) \neq 0$, and we give the details for what is probably the least trivial case. Suppose both $u^\alpha \neq u^2$ and $v^\alpha \neq v^2$. Then if either $P(0) \neq 0$ or $P(1) \neq 0$, we are done. So assume $P(0) = P(1) = 0$. Hence $u^2 v^\alpha = v^2 u^\alpha$ and $u^2 + v^2 = u^\alpha + v^\alpha$. We proved above that there must be some $c \in F$ such that $u \neq c \neq v$ and $c^\alpha \neq c^2$. Let $x = c$. Then $P(x) \neq 0$. This completes Case 1.

For Case 2, the trace of L_1 and L_2 is

$$\left\{[0\,,1]\,,[c\,,0\,,0]\right\} \cup \left\{\left[u\,,\frac{u^\alpha + c^\alpha}{u+c}\,,0\right] \middle| \, c \neq u \in F\right\}.$$

And the trace of $[0\,,1]$ and $[c\,,0\,,0]$ is

$$\left\{[0\,,0]\,,[c\,,0\,,1]\right\} \cup \left\{\left[v\,,\frac{v^\alpha + c^\alpha}{v+c}\,,1\right] \middle| \, c \neq v \in F\right\}.$$

For each line in the trace of L_1 and L_2 to be concurrent with each line in the trace of $[0\,,1]$ and $[c\,,0\,,0]$ it is easily seen to be necessary (and sufficient) that $u^2(c^\alpha + v^\alpha) + c^2(u^\alpha + v^\alpha) + v^2(u^\alpha + c^\alpha) = 0$ for all $u\,,v \in F$ with $u\,,c\,,v$ distinct. By Case 1 we are done. This completes the Proof of the Lemma.

To complete the proof of Theorem 3.1, let L_1 and L_2 be concurrent lines neither one of which meets $[\infty]$. Let x be any point of L_1 not on L_2 and not collinear with (∞). There is a unique line L_3 through (∞) which meets L_2. Then each line L_4 through x but different from L_1 and not meeting L_3 is a line for which L_4 meets L_1 and $(L_4\,,L_2)$ is an irregular pair by Lemma 3.2. Hence $[\infty]$ is a center of irregularity for the dual of $P(\alpha)$.

It is now clear that Theorem 2.6 applies to all $P(\alpha)$ and to their duals, provided $\alpha \neq 2$. It is also clear that (∞) is a pivotal point of $P(\alpha)$, so that the full collineation group G of $P(\alpha)$ must fix (∞), except when $\alpha = 2$. However, in general $[\infty]$ is not a pivotal line (i.e. a pivotal point of the dual of $P(\alpha)$), and there are cases in which $[\infty]$ is not fixed by G.

This completes the proof of our main results. In the next section we summarize a few of the known results concerning the available α.

4. The class of α's

Let S denote the set of permutations α of $F = GF(2^e)$ satisfying $0^\alpha = 0$, $1^\alpha = 1$, and $\dfrac{c_0^\alpha - c_1^\alpha}{c_0 - c_1} \neq \dfrac{c_0^\alpha - c_2^\alpha}{c_0 - c_2}$ for distinct $c_0\,,c_1\,,c_2$ in F. The exact size of S for given e seems to be a very difficult problem. For each integer i, $1 \leq i \leq e$, the automorphism $\beta : x \to x^{2^i}$ is in S if and only if g.c.d $(i\,,e) = 1$. In [9] it is proved that any additive map α in S must in fact be one of these automorphisms. Using this result we can prove the following:

THEOREM 4.1. *If $P(\alpha)$ has a regular point other than (∞), then $P(\alpha)$ is isomorphic to $P(\beta)$ for some automorphism $\beta \in S$.*

Proof: Theorem 3.2 of [8] says that for $\alpha \in S$, (∞) is the only regular point of $[\infty]$ except when α is additive (and hence an automorphism), in which case each point of $[\infty]$ is regular. If some point other than (∞) on $[0]$ is regular, then in $P(\alpha^*)$ all points of $[\infty]_{\alpha^*}$ are regular. Here α^* derives from α via (4) of [6], and $P(\alpha)$ is isomorphic to $P(\alpha^*)$. Hence to prove the theorem we may suppose that $P(\alpha)$ has a regular point of the form (u, v) (in the new notation of course). Then by Lemmas 2 and 3 of [6] we may assume $v = 0$. In particular, the pair $((u, 0), (0, 0, 1))$ is regular. The trace of $(u, 0)$ and $(0, 0, 1)$ is

$$T_1 = \{(u, 1), (0, 0, 0)\} \cup \{(x, \bar{x}, u^\alpha x) \mid 0 \neq x \text{ and } \bar{x} = x(x^{-1} + u^\alpha)^{\alpha^{-1}}\}.$$

Similarly, the trace of $(u, 1)$ and $(0, 0, 0)$ is

$$T_2 = \{(u, 0), (0, 0, 1)\} \cup \{(y, \bar{y}, u^\alpha y + 1) \mid y \neq 0, \bar{y} = y(y^{-1} + u^\alpha)^{\alpha^{-1}}\}.$$

Hence $(x, \bar{x}, u^\alpha x)$ must be collinear with $(y, \bar{y}, u^\alpha y + 1)$ for all $x, y \in F$. For $x \neq y$ this means $\left(\frac{\bar{x} + \bar{y}}{x + y}\right)^\alpha = (x + y)^{-1} + u^\alpha$. Hence

$$\bar{x} + \bar{y} = (x + y)[(x + y)^{-1} + u^\alpha]^{\alpha^{-1}} = \overline{x + y},$$

so that $x \to \bar{x}$ is an additive map. Using (4) we then have that

$$(\alpha^{-1})_{u\alpha} : x \to \frac{x[u + (u^\alpha + x^{-1})^{\alpha^{-1}}]}{u + (u^\alpha + 1)^{\alpha^{-1}}} = \left(\frac{u}{1 + u}\right)x + \left(\frac{1}{1 + u}\right)\bar{x}$$

is an additive map in S. Hence $(\alpha^{-1})_{u\alpha}$ must be some automorphism $\beta : x \to x^{2^i}$, g.c.d. $(i, e) = 1$. And the complete oval associated with α is projectively equivalent to that associated with β. Also, $P(\alpha)$ must be isomorphic to one of $P(\beta)$, $P(\beta^{-1})$, $P(1 - \beta)$. But β^{-1} is an automorphism. So the problem of which $P(\alpha)$ have regular points other than (∞) is reduced to determining what the regular points are of $P(\beta)$ and $P(1 - \beta)$, respectively. But this was resolved by Corollary 3.3 of [8] which states: *If* $\alpha \in S$ *is multiplicative,* $2 \neq \alpha$, *then* $P(\alpha)$ *has a line of regular points, either* $[\infty]$ *or* $[0]$, *according as* α *or* $\alpha^* = \frac{1}{\alpha - 1}$ *is an automorphism of* F. *If neither* α *nor* α^* *is an automorphism, then* (∞) *is the unique regular point of* $P(\alpha)$. Hence $P(1 - \beta)$ will have a line of regular points only if $1 - \beta$ or $(1 - \beta)^* = \frac{\beta - 1}{\beta}$ is an automorphism. Then $1 - \beta$ is an automorphism precisely when $\beta = 1/2$, in which case $1 - \beta = 1/2$. And $\frac{\beta - 1}{\beta}$ is an automorphism precisely when $\beta = 2$, in which case $(1 - \beta)^* = 1/2$. In any case, the proof of the Theorem is complete.

For each automorphism β there are three nonisomorphic quadrangles $P(\beta) \simeq P(\beta^*)$, $P(\beta^{-1}) \simeq P((\beta^{-1})^*)$, and $P(1 - \beta) \simeq P\left(\frac{\beta - 1}{\beta}\right)$. By Theorem 1.2 of [7] the dual of $(1 - \beta)$ (in case $1 - \beta$ is not an automorphism) yields an additional distinct quadrangle. Hence there are at least $2 \cdot \varphi(e)$ distinct quadrangles when $e \geq 3$. In some cases there are more. Let us say that a multiplicative α in S is *metamorphic* if it is among $(\beta^{-1})^*$, β^{-1}, β, β^*, $(\beta^*)^{-1}$, $((\beta^*)^{-1})^*$, for some automorphism $\beta \in S$. We close with two examples of multiplicative α which are not metamorphic.

Example 1. Let α be the permutation of $\mathrm{F} = \mathrm{GF}(2^5)$ defined by $\alpha = 6 : x \to x^6$. Then $\alpha \in \mathrm{S}$, and $\alpha^* = \alpha^{-1} = 26$, $26^* = 26^{-1} = 6$. Hence $P(6) \simeq P(26)$, and 6 is not metamorphic. We have checked that 6 and 26 are the only nonmetamorphic multiplicative α in S (as noted in [7]).

Example 2. Let α be the permutation of $\mathrm{F} = \mathrm{GF}(2^7)$ defined by $\alpha = 20 : x \to x^{20}$. Then $\alpha \in \mathrm{S}$ and $\alpha^* = \alpha^{-1} = -19$, $(-19)^* = (-19)^{-1} = 20$. Here also it turns out that the only nonmetamorphic multiplicative $\alpha \in \mathrm{S}$ are 20 and -19.

It is a curious fact that for $e = 5$ and $e = 7$ the only nonmetamorphic multiplicative $\alpha \in \mathrm{S}$ are precisely those α satisfying $\alpha^2 - \alpha + 1 \equiv 0 \pmod{2^e - 1}$. However, we resist the temptation to make a conjecture on the basis of such minimal evidence.

The following contribution to the study of generalized quadrangles has just come to our attention: J. THAS, *On 4-gonal configurations*, «Geometriae Dedicata», to appear. It seems unclear to us whether or not the constructions given in that paper will actually yield new quadrangles, but it certainly seems possible. Hence the phrase "all known quadrangles" used in the present work should be understood to mean just those included in the discussion given in this paper. Also, we would remind the reader of the following article which is of considerable interest especially in connection with the last section of the present work: B. SEGRE and U. BARTOCCI, *Ovali ed altre curve nei piani di Galois di caratteristica due*, «Acta Arith.», *18*, 423–449 (1971).

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