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Operators of ces—p type

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Analisi funzionale. — *Operators of ces–p type*. Nota di GHEORGHE CONSTANTIN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si studia una nuova classe di operatori appartenenti alla classe degli operatori di *Pietsch* e connessi con gli operatori di *Cesáro*.

1. Let X and Y be normed linear spaces and

$$\alpha_n(\mathrm{T}) = \inf\{\| \mathrm{T} - \mathrm{A} \| : \mathrm{A} \in \mathfrak{R}_n(\mathrm{X}, \mathrm{Y})\}$$

where $\mathfrak{R}_n(\mathrm{X}, \mathrm{Y}) = \{\mathrm{A}, \mathrm{A} : \mathrm{X} \to \mathrm{Y}, \dim \mathrm{R}(\mathrm{A}) \leq n\}$, $\mathrm{R}(\mathrm{A}) = \{y \in \mathrm{Y} : \exists x \in \mathrm{X}, \mathrm{A}x = y\}$.

A. Pietsch [3] has introduced the notion of operator of l^p type as an operator $\mathrm{T} \in \mathfrak{L}(\mathrm{X}, \mathrm{Y})$ for which

$$\sum_{n=0}^{\infty} [\alpha_n(\mathrm{T})]^p < \infty .$$

For the Cesàro operator C_0 defined on a sequence space by $(\mathrm{C}_0 x)_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k$, it is proved in [1] that $\mathrm{C}_0 \in \mathfrak{L}(l^2, l^2)$ and is a hyponormal operator.

In this Note we introduce a new class of operators using the Cesàro sequence spaces ces–p and we give some properties for this class of operators suggested by [3] and [1].

2. In " Nieuw Arch. Wiskunde ", *17* (1), 1971, p. 70, are defined the Cesàro sequence spaces ces–p as the spaces of all numerical sequences $x = (x_0, x_1, \cdots, x_n, \cdots)$ with finite norms

$$|x|_p = \left[\sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k|\right)^p\right]^{1/p}, \qquad 1 \leq p < \infty$$

$$|x|_\infty = \sup_{n \geq 0} \left\{\frac{1}{n+1} \sum_{k=0}^{n} |x_k|\right\}.$$

We give

DEFINITION 2.1. *Let* X *and* Y *be normed linear spaces. An operator* $\mathrm{T} \in \mathfrak{L}(\mathrm{X}, \mathrm{Y})$ *is called of ces–p type if* $\{\alpha_n(\mathrm{T})\}_{n=0}^{\infty}$ *is an element of ces–p space.*

PROPOSITION 2.1. *The set of operators of ces–p type is a linear space.*

Proof. Let S and T be operators of ces–p type. Since $\{\alpha_k(\mathrm{T})\}_{k=0}^{\infty}$ is a decreasing sequence for every operator $\mathrm{T} \in \mathfrak{L}(\mathrm{X}, \mathrm{Y})$ and for all nonnegative

(*) Nella seduta del 16 giugno 1972.

integers r, s we have $\alpha_{r+s}(S+T) \leq \alpha_r(T) + \alpha_s(S)$, which gives

$$\sum_{n=0}^{\infty} \left[\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k(S+T)\right]^p \leq \sum_{n=0}^{\infty} \left[\frac{2}{n+1} \sum_{n=0}^{[n/2]} \alpha_{2k}(S+T)\right]^p \leq$$

$$\leq \sum_{n=0}^{\infty} \left\{\frac{2}{n+1} \sum_{k=0}^{[n/2]} [\alpha_k(S) + \alpha_k(T)]\right\}^p \leq$$

$$\leq 2^p \tau_p \left\{\sum_{n=0}^{\infty} \left[\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k(S)\right]^p + \sum_{n=0}^{\infty} \left[\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k(T)\right]^p\right\}$$

because for $a, b \geq 0$ is satisfied the inequality

$$(*) \qquad (a+b)^p \leq \tau_p (a^p + b^p) \quad , \quad \tau_p = \max(2^{p-1}, 1) .$$

Therefore $S+T$ is of ces–p type and it is easy to see that λT is of ces–p type for all scalars λ.

The connection between the class of operators of ces–p type and l^p type is given by

PROPOSITION 2.2. *Every operator* T *of ces–p type is an operator of l^p type.*

The proof follows from the relation

$$\sum_{n=0}^{\infty} \left[\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k(T)\right]^p \geq \sum_{n=0}^{\infty} \left[\frac{(n+1)\,\alpha_n(T)}{n+1}\right]^p = \sum_{n=0}^{\infty} [\alpha_n(T)]^p .$$

COROLLARY. *The class of operators of ces–1 type contains only the operator* $T = 0$.

In a similar way as in [3] we define an application $\beta_p : \text{ces–}p \to R$ by

$$\beta_p(T) = \left\{\sum_{n=0}^{\infty} \left[\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k(T)\right]^p\right\}^{1/p}$$

which possesses the following properties:

1) $\beta_p(T) \geq 0$;
2) $\beta_p(T) = 0$ implies $T = 0$;
3) $\beta_p(\lambda T) = |\lambda| \, \beta_p(T)$ for all scalars λ;
4) there exists a number $\sigma_p \geq 1$ such that

$$\beta_p(S+T) \leq \sigma_p [\beta_p(S) + \beta_p(T)]$$

for all operators S, T of ces–p type.

These properties show that $\beta_p(T)$ is a quasi-norm which defines on the space of operators of ces–p type a metrisable topology.

PROPOSITION 2.3. *Let* $\{T_n\}_{n=1}^{\infty}$ *a fundamental sequence of operators of ces–p type such that there exists* $T \in \mathfrak{L}(X, Y)$ *with* $\lim_{n\to\infty} T_n x = Tx$ *for all* $x \in X$. *Then* T *is an operator of ces–p type and* $\beta_p - \lim_{n\to\infty} T_n = T$.

Proof. Since $\alpha_0(S) = \| S \| \leq \beta_p(S)$ for every operator S of ces–p type we have that $\{T_n\}_{n=1}^{\infty}$ is a fundamental sequence in $\mathfrak{L}(X, Y)$. From the fact that $\lim\limits_{n\to\infty} T_n x = Tx$ for all $x \in X$, we conclude that $\{T_n\}_{n=1}^{\infty}$ converges to T in $\mathfrak{L}(X, Y)$.

It is known that for all $S, T \in \mathfrak{L}(X, Y)$ and $s \in N$ we have $|\alpha_s(S) - \alpha_s(T)| \leq \leq \| S - T \|$ [3] and therefore

$$|\alpha_s(T - T_n) - \alpha_s(T_m - T_n)| \leq \| T - T_m \|$$

which gives

$$\lim_{m\to\infty} \alpha_s(T_m - T_n) = \alpha_s(T - T_n) .$$

But for each $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$\beta_p(T_m - T_n) = \left\{ \sum_{r=0}^{\infty} \left[\frac{1}{r+1} \sum_{k=0}^{r} \alpha_k(T_m - T_n) \right]^p \right\}^{1/p} \leq \varepsilon , \qquad n, m \geq n_0(\varepsilon)$$

and for $m \to \infty$ we obtain

$$\beta_p(T - T_n) = \left\{ \sum_{r=0}^{\infty} \left[\frac{1}{r+1} \sum_{k=0}^{r} \alpha_k(T - T_n) \right]^p \right\}^{1/p} , \qquad n \geq n_0(\varepsilon) .$$

It follows that $T - T_n$ is of ces–p type and therefore T is an operator of ces–p type and

$$\beta_p - \lim_{n\to\infty} T_n = T$$

PROPOSITION 2.4. *If* X *is a normed space and* Y *a Banach space then the space of operators of ces–p type is complete.*

Let $T \in \mathfrak{R}_q$. Then

$$\sum_{n=0}^{\infty} \left[\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k(T) \right]^p = \sum_{n=0}^{q} \left[\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k(T) \right]^p + \sum_{q+1}^{\infty} \left[\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k(T) \right]^p$$

and since

$$\left[\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k(T) \right]^p = \frac{\alpha_0(T) + \cdots + \alpha_q(T)}{(n+1)^p} , \qquad \forall n \geq q$$

we have that this series converges at the same time with $\sum\limits_{n=0}^{\infty} \frac{1}{(n+1)^p}$. Therefore every operator of finite rank is an operator of ces–p type for all $p > 1$.

PROPOSITION 2.5. *Let* X, Y, Z *be normed linear spaces. If* $T \in \mathfrak{L}(X, Y)$ *and* $S : Y \to Z$ *is of ces–p type then* ST *is of ces–p type and*

$$\beta_p(ST) \leq \|T\| \, \beta_p(S) .$$

Also if $T : X \to Y$ is of ces–p type and $S \in \mathfrak{L}(Y, Z)$ then ST is of ces–p type and

$$\beta_p(ST) \leq \|S\| \, \beta_p(T) .$$

We recall [5] that a set $\mathfrak{A} \subset \mathfrak{L}(X, Y)$ is a β–normed ideal if

1) $\mathfrak{A}$ is a linear subspace of $\mathfrak{L}(X, Y)$;

2) on $\mathfrak{A}$ is defined a quasi-norm β;

3′) if $T \in \mathfrak{A}(X, Y)$ and $S \in \mathfrak{L}(Y, Z)$ then $ST \in \mathfrak{L}(X, Z)$ and
$$\beta(ST) \leq \|S\| \, \beta(T);$$

3″) if $T \in \mathfrak{L}(X, Y)$ and $S \in \mathfrak{A}(Y, Z)$ then $ST \in \mathfrak{A}(X, Z)$ and
$$\beta(ST) \leq \|T\| \, \beta(S) .$$

A β–normed ideal is called complete if is complete relatively to the quasi-norm β.

From the Propositions 2.1, 2.4 and 2.5 we obtain

PROPOSITION 2.6. *The set of operators of ces–p type is a completely β–normed ideal.*

The following proposition gives a method for obtaining examples of operators of ces–p type.

PROPOSITION 2.7. *Let m be the space of bounded sequences and c the space of convergent sequences. An operator $T \in \mathfrak{L}(m, m)$ of the form*

$$T\{x_i\} = \{a_i x_i\} , \qquad \{a_i\} \in c$$

is an operator of ces–p type if and only if

$$\beta_p(T) = \left\{ \sum_{n=0}^{\infty} \left[\frac{1}{n+1} \sum_{k=0}^{n} |a_k| \right]^p \right\}^{1/p} < \infty .$$

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