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On Comparison Theorems for Matrix Inequalities with Mixed Boundary Conditions

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Analisi matematica. — *On Comparison Theorems for Matrix Inequalities with Mixed Boundary Conditions* (*). Nota di EUTIQUIO C. YOUNG, presentata (**) dal Socio M. PICONE.

RIASSUNTO. — In questa Nota si estendono i risultati conseguiti da Swanson nel confronto fra due sistemi di equazioni lineari autoaggiunte con condizioni miste al contorno a sistemi le cui matrici sono quasi lineari e non autoaggiunte.

1. INTRODUCTION

The Sturmian comparison theorem of Kuks [1] for linear strongly elliptic systems was recently generalized by Kreith [2], [3] by introducing more general boundary conditions, and by Kreith and Travis [4]. Kreith's technique was based on a generalization of a Picone identity. On the other hand, using a variational approach Swanson [5] extended Kuks result to quasilinear selfadjoint elliptic systems and sharpened Kuks' original theorem. Swanson's work was subsequently carried over to nonselfadjoint quasilinear matrix inequalities by Noussair [6] and thereby derived some oscillation and nonoscillation theorems. More recently, Swanson [7] improved a result of [4] and stated a corresponding result for linear selfadjoint systems with mixed boundary conditions. The purpose of this paper is to extend a result of [3] and [7] to quasilinear nonselfadjoint matrix inequalities. The result generalizes Theorem 1 of [6] in that it deals with more general boundary conditions.

Let K and L denote the elliptic differential operators defined by

$$\mathrm{K}u = -\mathrm{D}_i\,[a_{ij}(x\,,u)\,\mathrm{D}_j u] + 2\,b_i(x\,,u)\,\mathrm{D}_i u + c(x\,,u)\,u \tag{1}$$

and

$$\mathrm{LV} = -\mathrm{D}_i\,[\mathrm{A}_{ij}(x\,,\mathrm{V})\,\mathrm{D}_j\mathrm{V}] + 2\,\mathrm{B}_i(x\,,\mathrm{V})\,\mathrm{D}_i\mathrm{V} + \mathrm{C}(x\,,\mathrm{V})\,\mathrm{V} \tag{2}$$

$$x = (x_1\,,\cdots,x_n)\quad,\quad \mathrm{D}_i = \frac{\partial}{\partial x_i}\,,\qquad i = 1\,,\cdots,n$$

respectively, for $x \in \mathrm{R}$, $u \in \mathrm{H}$, $\mathrm{V} \in \mathrm{H}^m$ where R is a nonempty, regular, bounded domain in n-dimensional Euclidean space E^n with smooth boundary $\partial\mathrm{R}$. H is a domain in E^m containing the origin, H^m is the set of $m\times m$ matrices, and u is a m-vector function of class $\mathrm{C}^2(\mathrm{R}) \cap \mathrm{C}^1(\bar{\mathrm{R}})$ with range in H. The repeated indices are to be summed from 1 to n. The coefficients a_{ij}, b_i, and c are real $m\times m$ matrix functions of class $\mathrm{C}^1(\bar{\mathrm{R}}\times\mathrm{H})$. Analogous conditions

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are satisfied by the coefficients A_{ij}, B_i, and C in $\bar{R} \times H^m$. Further, the $mn \times mn$ matrix $(A_{ij}(x, V))$ is symmetric and positive definite in $\bar{R} \times H^m$.

The conclusion of the comparison theorems given below concerns $m \times m$ matrices $V \in C^2(R) \cap C^1(\bar{R})$ which are "prepared" in the sense that

$$V^T A_{ij}(x, V) D_j V, \qquad i = 1, \cdots, n \tag{3}$$

is symmetric with V^T denoting the transposed of V.

2. COMPARISON THEOREMS

Since the $mn \times mn$ matrix $(A_{ij}(x, V))$ is positive definite in $R \times H^m$, a diagonal matrix $G(x, V) = (g_{ii}(x, V))$ can be constructed such that the quadratic form

$$\begin{aligned} Q(u, V) = (VD_i u)^T A_{ij}(x, V) VD_j u + 2(Vu)^T B_i(x, V) VD_i u \\ + (Vu)^T C(x, V) Vu \end{aligned} \tag{4}$$

is positive semidefinite in $R \times H^m$. This can be done by using a criterion of Gantmacher [8] and an inductive argument.

THEOREM 1. *Let* V *be a prepared matrix satisfying*

$$\begin{cases} V^T LV \geq 0 & (\textit{positive semidefinite}) \textit{ in } R \\ \nu_i A_{ij}(x, V) D_j V + S(x) V = 0 & \textit{on } \Gamma_1 \subset \partial R \end{cases} \tag{5}$$

where S *is a* $m \times m$ *matrix function continuous on* Γ_1 *and* $(\nu_1, \cdots, \nu_n)$ *denotes the outward unit normal vector on* ∂R. *If there exists a nontrivial vector function* $u \in C^2(R) \cap C^1(\bar{R})$ *satisfying*

$$\begin{cases} u^T K u \leq 0 & \textit{in } R \\ \nu_i a_{ij}(x, u) D_j u + s(x) u = 0 & \textit{on } \Gamma_1 \\ u = 0 & \textit{on } \Gamma_2, \ \Gamma_1 \cup \Gamma_2 = \partial R \end{cases} \tag{6}$$

such that

$$\begin{aligned} F(u, V) \equiv \int_R \{ (D_i u)^T [a_{ij}(x, u) - A_{ij}(x, V)] D_j u \\ + 2u^T [b_i(x, u) - B_i(x, V)] D_i u \\ + u^T [c(x, u) - C(x, V) - G(x, V)] u \} dx \\ + \int_{\Gamma_1} u^T [s(x) - S(x)] u \, d\sigma > 0 \end{aligned} \tag{7}$$

then V *is singular at some point in* $\bar{R}$.

Proof. The proof depends on the following identity

$$u^{\mathrm{T}}\,\mathrm{K}u - u^{\mathrm{T}}\,(\mathrm{LV})\,\mathrm{V}^{-1}\,u$$

$$\begin{aligned}
&= -\mathrm{D}_i\,[u^{\mathrm{T}}\,a_{ij}(x\,,u)\,\mathrm{D}_j\,u - u^{\mathrm{T}}\mathrm{A}_{ij}(x\,,\mathrm{V})\,(\mathrm{D}_j\,\mathrm{V})\,\mathrm{V}^{-1}\,u] \\
&\quad + (\mathrm{D}_i\,u)^{\mathrm{T}}\,[a_{ij}(x\,,u) - \mathrm{A}_{ij}(x\,,\mathrm{V})]\,\mathrm{D}_j\,u + 2\,u^{\mathrm{T}}\,[b_i(x\,,u) - \mathrm{B}_i(x\,,\mathrm{V})]\,\mathrm{D}_i\,u \\
&\quad + [\mathrm{D}_i\,u - (\mathrm{D}_i\,\mathrm{V})\,\mathrm{V}^{-1}\,u]^{\mathrm{T}}\,\mathrm{A}_{ij}(x\,,\mathrm{V})\,[\mathrm{D}_j\,u - (\mathrm{D}_j\,\mathrm{V})\,\mathrm{V}^{-1}\,u] \\
&\quad + 2\,u^{\mathrm{T}}\,\mathrm{B}_i(x\,,\mathrm{V})\,[\mathrm{D}_i\,u - (\mathrm{D}_i\,\mathrm{V})\,\mathrm{V}^{-1}\,u] + u^{\mathrm{T}}\,(\mathrm{G}(x\,,\mathrm{V}))\,u \\
&\quad + u^{\mathrm{T}}\,[c(x\,,u) - \mathrm{C}(x\,,\mathrm{V}) - \mathrm{G}(x\,,\mathrm{V})]\,u
\end{aligned}$$

which is readily established by making use of the symmetric property of (3) and (A_{ij}) (see [4]). Suppose V is nonsingular in $\overline{\mathrm{R}}$. Then there exists a unique $w \in \mathrm{C}^1(\overline{\mathrm{R}})$ such that $u = \mathrm{V}w$ or $w = \mathrm{V}^{-1}u$ in $\overline{\mathrm{R}}$. Substituting this in (8) and integrating over R, making use of the boundary conditions satisfied by u and V, we obtain by Green's theorem

$$\int_{\mathrm{R}} [u^{\mathrm{T}}\,\mathrm{K}u - w^{\mathrm{T}}\,\mathrm{V}^{\mathrm{T}}\,\mathrm{LV}w - \mathrm{Q}(w\,,\mathrm{V})]\,\mathrm{d}x = \mathrm{F}(u\,,\mathrm{V}) \tag{9}$$

where $\mathrm{Q}(w\,,\mathrm{V})$ is given in (4) and $\mathrm{F}(u\,,\mathrm{V})$ in (7). By hypothesis, the left hand side in (9) is nonpositive while the right hand side is positive. This contradiction shows that V must vanish at some point in $\overline{\mathrm{R}}$.

We remark that if G is determined such that the quadratic form (4) is positive definite, then the condition (7) can be replaced by $\mathrm{F}(u\,,\mathrm{V}) \geq 0$. Indeed, in such case equation (9) implies that both sides of the equation must vanish. Since each term in the integrand on the left hand side of (9) is nonpositive, we conclude in particular that

$$\int_{\mathrm{R}} \mathrm{Q}(w\,,\mathrm{V})\,\mathrm{d}x = 0 \tag{10}$$

By the positive definiteness of Q, (10) holds if and only if $w \equiv 0$ in $\overline{\mathrm{R}}$, which implies that $u \equiv \mathrm{V}w \equiv 0$ in $\overline{\mathrm{R}}$. This contradicts the fact that u is a nontrivial vector function; hence the conclusion of the theorem.

A variational type theorem corresponding to Theorem 1 is given below.

THEOREM 2. *Let* V *be a prepared matrix satisfying* (5). *If there exists a nontrivial function* $u \in \mathrm{C}^2(\mathrm{R}) \cap \mathrm{C}^1(\overline{\mathrm{R}})$ *vanishing on* Γ_2 *such that*

$$\begin{aligned}
\int_{\mathrm{R}} [(\mathrm{D}_i u)^{\mathrm{T}}\,\mathrm{A}_{ij}(x\,,\mathrm{V})\,\mathrm{D}_j\,u + 2\,u^{\mathrm{T}}\,\mathrm{B}_i(x\,,\mathrm{V})\,\mathrm{D}_i\,u \\
+ u^{\mathrm{T}}\,(\mathrm{C}(x\,,\mathrm{V}) + \mathrm{G}(x\,,\mathrm{V}))\,u]\,\mathrm{d}x \\
+ \int_{\Gamma_1} u^{\mathrm{T}}\,\mathrm{S}(x)\,u\,\mathrm{d}\sigma < 0
\end{aligned}$$

then V *is singular in* $\overline{\mathrm{R}}$.

The proof is similar to that of Theorem 1, making use of the identity (8) with the coefficients a_{ij}, b_i, and c all set equal to zero.

3. SELFADJOINT CASE

In the case that $b_i = B_i = 0$ $(i = 1, \cdots, n)$ so that the operators K and L become

$$K^* u = - D_i [a_{ij}(x, u) D_j u] + c(x, u) u$$

and

$$L^* V = - D_i [A_{ij}(x, V) D_j V] + C(x, V) V$$

respectively, we choose $G \equiv 0$ in (4). Then we have the following generalization of a result of [3] and [7].

THEOREM 3. *Let* V *be a prepared matrix satisfying* (5) *with* L *replaced by* L^*. *If there exists a nontrivial vector function* u *satisfying* (6), *with* K *replaced by* K^*, *such that*

$$\int_R \{(D_i u)^T [a_{ij}(x, u) - A_{ij}(x, V)] D_j u + u^T [c(x, u) - C(x, V)] u\} dx + \int_{\Gamma_1} u^T [s(x) - S(x)] u \, d\sigma \geq 0$$

then V *is singular at some point in* $\bar{R}$.

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