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Function Algebras over Valued Fields and Measures. Nota III

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Algebra topologica. — *Function Algebras over Valued Fields and Measures.* Nota III di EDWARD BECKENSTEIN, GEORGE BACHMAN e LAWRENCE NARICI, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Si studiano gli anelli di funzioni continue C (T , F) definite su uno spazio topologico di Hausdorff O–dimensionale a valori in un campo completo F non archimedeo di rango uno, a valutazione non banale.

SECTION ZERO. INTRODUCTION

In this paper we consider rings of continuous functions C (T , F) taking a O–dimensional Hausdorff topological space T into a complete nonarchimedean rank one nontrivially valued field F. We analyse these structures in the spirit of [5] utilizing the notion of 0–1 measures and answer a number of questions raised in [1].

Principal among our results (Theorem 1) is that if F is a complete discretely valued field whose residue class field has nonmeasurable cardinal, then a maximal ideal M of C (T , F) is the kernel of a homomorphism of C (T , F) into F if and only if the collection $Z(M) = \{f^{-1}(O) \mid f \in M\}$ is closed with respect to the formation of denumerable intersections. This is an analog of a result of Hewitt [3, page 75] in which the underlying field F is the real number R. Of interest is the fact that properties of R which are useful in proofs of this result (e.g. R is an ordered field, $\sqrt{-1} \notin R$, R is locally compact) are not properties of the fields considered here. The broad category of discretely valued fields whose residue class fields have nonmeasurable cardinal includes the local fields. We show that the statement of Theorem 1 is true for all complete and discretely valued fields if and only if all cardinals are nonmeasurable. Thus we obtain a functional-analytic equivalent to the long standing conjecture that all cardinals are nonmeasurable.

Throughout the paper T will denote a 0–dimensional Hausdorff topological space, $\mathcal{S}$ the base for the topology consisting of all closed and open sets, and F a complete nonarchimedean rank one nontrivially valued field. The algebra of all continuous functions taking T into F will be denoted by C (T , F), the algebra of bounded continuous functions taking T into F by $C^*(T, F)$. The F–valued characteristic function of a subset E of T is denoted by k_E.

In this Note III only sections 1 and 2 appear. Sections 3, 4 and 5 will appear in Note IV.

(*) Nella seduta del 16 giugno 1972.

1. MAXIMAL IDEALS AND ZERO SETS

In this section we consider the nature of zero sets $z(f) = f^{-1}(O)$ for some $f \in C(T, F)$ and their relationship to the maximal ideals of $C(T, F)$ (Proposition 4). The collection of all zero sets of T will be shown (Proposition 1) to be independent of the field F and will be referred to as $\mathfrak{z}$.

DEFINITION 1. *A subset* E *of* T *is a* C_δ *set if there exists a denumerable family* (S_n) *of clopen (closed and open) sets such that* $E = \bigcap_{n=1}^{\infty} S_n$.

PROPOSITION 1. *A subset* E *of* T *is a zero set if and only if* E *is a* C_δ *set.*

Proof. If $E = z(f)$, then $E = \bigcap_{n=1}^{\infty} \{t \in T \mid |f(t)| < 1/n\}$. Conversely, if $E = \bigcap S_n$, then choosing $\alpha \in F$ such that $|\alpha| < 1$ and setting $f = \sum_{n=1}^{\infty} \alpha^n k_{CS_n}$, we find that $z(f) = E$.

PROPOSITION 2. *If* E *is a* C_δ *set there exists* $f \in C^*(T, F)$ *such that* $E = z(f)$.

Proof. We refer to the function f constructed in the proof of Proposition 1, and note that f is a bounded function.

PROPOSITION 3. *Disjoint* C_δ *sets can be separated by disjoint clopen sets.*

Proof. By Proposition 1 we may take F to be a field such that $\sqrt{-1} \notin F$. Suppose $z(f) \cap z(g) = \varnothing$ for some $f, g \in C(T, F)$. Then let $h = f^2/(f^2 + g^2)$ and we see that $z(f) \subset \{t \in T \mid |h(t)| < 1\}$ while $z(g) \subset \{t \in T \mid |h(t)| \geqq 1\}$.

DEFINITION 2. *A z-filter* $\mathfrak{Z}$ *is a collection of nonempty zero sets stable under finite intersections such that if* $E \in \mathfrak{Z}$ *and* $E \subset K \in \mathfrak{z}$, *then* $K \in \mathfrak{Z}$.

As an intersection of two zero (C_δ) sets is a zero (C_δ) set, it can readily be shown that every z-filter can be extended to a maximal z-filter (z-ultrafilter). We wish to show now that for any field F, there is a 1-1 correspondence between the z-ultrafilters and the maximal ideals of $C(T, F)$ where a z-ultrafilter $\mathfrak{Z}$ is shown to be the collection of zero sets of the functions in some maximal ideal $M \subset C(T, F)$, and this establishes the correspondence. To demonstrate this we first prove three preliminary Lemmas.

LEMMA 1. *Let* I *be an ideal in* $C(T, F)$, *and* S *a clopen subset of* T. *If* $S = z(f)$ *for some* $f \in I$, *then* $k_{CS} \in I$.

Proof. Consider the function $g(t) = \begin{cases} 0, & t \in S \\ f(t)^{-1}, & t \in CS. \end{cases}$

Clearly $gf = k_{CS} \in I$.

LEMMA 2. *If* I *is a proper ideal in* $C(T, F)$ *and* $z(f), z(g) \in Z(I) = \{z(f) \mid f \in I\}$, *then* $z(f) \cap z(g) \neq \varnothing$.

Proof. If $z(f) \cap z(g) = \varnothing$, then by Proposition 3 there exists a clopen set S such that $z(f) \subset S$ while $z(g) \subset CS$. Since I is an ideal it follows that S and CS both belong to Z (I) and therefore by Lemma 1, k_S and k_{CS} both belong to I. But then I is not a proper ideal.

LEMMA 3. *If* M *is a maximal ideal and* $z(f), z(g) \in Z(M)$, *then* $z(f) \cap z(g) \in Z(M)$.

Proof. First we note that if $z(x) = z(h)$ for some $h \in M$, then $x \in M$. This follows as a consequence of Lemma 2, for clearly the ideal generated by $M \cup \{x\}$ is a proper ideal.

We may assume that $f, g \in M$. There exists a pair of sequences of clopen sets (W_i) and (S_i) such that $z(f) = \bigcap_{i=1}^{\infty} W_i$ and $z(g) = \bigcap_{i=1}^{\infty} S_i$. We choose $\alpha \in F$ such that $|\alpha| < 1$ and consider the continuous functions $f' = \sum_{i=1}^{\infty} \alpha^{2i} k_{CW_i}$ and $g' = \sum_{i=1}^{\infty} \alpha^{2i+1} k_{CS_i}$. Clearly $f', g' \in M$ and $z(f' + g') = z(f) \cap z(g)$.

PROPOSITION 4. *There exists a* 1–1 *correspondence between maximal ideals of* $C(T, F)$ *and z–ultrafilters where for each maximal ideal* M *the mapping*

$$M \to \{z(f) \mid f \in M\}$$

establishes the correspondence.

Proof. Based on Lemma 3 and the fact that a finite intersection of zero (C_δ) sets is a zero (C_δ) set, we may use a standard argument for the proof as can be found in [2].

2. ZERO–ONE MEASURES ON $\mathcal{S}$ AND THEIR EXTENSIONS

In this section we explore the relationships between certain types of 0–1 measures on $\mathcal{S}$ and maximal ideals of $C(T, F)$. We find that there is a 1–1 correspondence between maximal ideals M of $C(T, F)$ and these special types of 0–1 measures.

DEFINITION 3. *A monotone* 0–1 *measure on* $\mathcal{S}$ *is a map* $\mu : \mathcal{S} \to F$ *such that for all* $S, W \in \mathcal{S}$

(1) $\mu(S) \in \{0, 1\}$.

(2) *If* $S \cap W = \varnothing$, *then* $\mu(S \cup W) = \mu(S) + \mu(W)$.

(3) *If* $S \subset W$ *and* $\mu(W) = 0$, *then* $\mu(S) = 0$.

(4) $\mu(T) = 1$.

We note that if char $F \neq 2$ then (3) follows from (1) and (2). In this case it hardly matters to what field the values 0 and 1 belong, as one cannot distinguish between the properties of a 0–1 measure taking values in one such field or another. In adding property (3) to the definition of our measure we ensure that μ will be indistinguishable from a measure whose values are taken in a field of characteristic $\neq 2$. As a result of this, it will not be necessary to restrict the characteristic of the field F in any of the results to follow.

PROPOSITION 5. *There is a* 1–1 *correspondence between the monotone* 0–1 *measures* μ *on* $\mathcal{S}$ *and the maximal ideals of* $C(T, F)$ *where the mapping*

$$M \to \mu$$

with $\mu(S) = 1$ *if and only if* $S \in Z(M)$ *establishes the correspondence.*

Proof. Given a maximal ideal M let us define μ with $\mu(S) = 1$ if and only if $S \in Z(M)$. As a result of the properties of the z–ultrafilter $Z(M)$, it is clear that μ is a monotone 0–1 measure on $\mathcal{S}$.

Conversely, if μ is a monotone 0–1 measure on $\mathcal{S}$, let us consider the ideal

$$I = [\{k_{CS} \mid S \in \mathcal{S}, \mu(S) = 1\}].$$

Since μ is monotone (i.e., property (3) applies), then if $\mu(S_i) = 1$ for $i = 1, \cdots, n$, it follows that $\mu\left(\bigcap_{i=1}^{n} S_i\right) = 1$ and I is a proper ideal. Extending I to a maximal ideal M, it can be shown that $\mu(S) = 1$ if and only if $S \in Z(M)$. The 1–1 correspondence is now clear.

Under the correspondence of Proposition 5 we refer to the maximal ideal generated by μ as M^{μ}.

DEFINITION 4. *A monotone* 0–1 *measure* μ *on* $\mathcal{S}$ *is said to be* σ–*smooth if given a denumerable descending sequence* S_n *of clopen sets such that* $\bigcap_{n=1}^{\infty} S_n = \varnothing$, *then* $\mu(S_n) \to 0$.

Of course in the above definition we see that $\mu(S_n)$ must ultimately become equal to 0.

PROPOSITION 6. *If* M^{μ} *is the kernel of a homomorphism of* $C(T, F)$ *into* F *then* μ *is* σ–*smooth.*

Proof. Suppose that M^{μ} is the kernel of a homomorphism h. Let S be a clopen set such that $S = \bigcup_{i=1}^{\infty} S_i$ and $S_i \cap S_j = \varnothing$ for all i, j. To show that μ is σ–smooth it is necessary and sufficient to prove that $\mu(S) = \sum_{i=1}^{\infty} \mu(S_i)$. We may consider two cases: $\mu(S_i) = 0$ for all i, and $\mu(S_i) = 1$ for some i.

Assuming first that $\mu(S_i) = 0$ for all i, take $\alpha \in F$ such that $0 < |\alpha| < 1$ and consider $f = \sum_{i=1}^{\infty} \alpha^i k_{S_i} + k_{CS}$. Since $z(f - h(f) k_T) = W \in Z(M^{\mu})$ and

$W = CS$ or $W = S_{i_0}$ for some integer i_0, it follows from the fact that $\mu(W) = 1$ and $\mu(S_{i_0}) = 0$ that $W = CS$ and therefore that $\mu(S) = 0$.

The case where $\mu(S_i) = 1$ for some i is trivial.

We now show that any monotone measure μ can be extended to the algebra $\mathcal{E}$ of subsets of T generated by $\mathfrak{z}$.

DEFINITION 4. *Let μ be a monotone 0–1 measure on $\mathcal{S}$ and*

$$a(\mu) = \left\{ E \subset T \,\middle|\, \begin{array}{ll} E \supset z(f), & \textit{some } f \in M^{\mu} \\ CE \supset z(f), & \textit{some } f \in M^{\mu} \end{array} \right\}.$$

If $E \in a(\mu)$ and $E \supset z(f)$ for some $f \in M^{\mu}$, we define $\hat{\mu}(E) = 1$. Otherwise $\hat{\mu}(E) = 0$.

It is clear that $a(\mu)$ is an algebra of sets containing $\mathfrak{z}$ and that $\hat{\mu}$ is a monotone 0–1 measure on $a(\mu)$.

PROPOSITION 7. *There is a 1–1 correspondence between maximal ideals M^{μ} of $C(T, F)$ and monotone 0–1 measures μ on $\mathcal{S}$ where the relationship*

$$\mu \to M^{\mu} = \{ f \in C(T, F) \mid \hat{\mu}(z(f)) = 1 \}$$

establishes the correspondence.

Proof. We refer to the properties of $Z(M^{\mu})$ (Proposition 4) and the properties of the extension $\hat{\mu}$ of μ for the proof of this result.

LEMMA 4. *μ is σ–smooth on $\mathcal{S}$ if and only if the extension $\hat{\mu}$ of μ is σ–smooth on $a(\mu)$.*

Proof. Let us suppose that μ is σ–smooth on $\mathcal{S}$ and that $E_i \in a(\mu)$ with $E_i \downarrow \varnothing$. We wish to show that it is not possible to have $\hat{\mu}(E_i) = 1$ for all i. If this were so then $E_i \supset z(f_i)$ where $\hat{\mu}(z(f_i)) = 1$ for each i. As each $z(f_i)$ is a C_δ set, we can then construct a sequence of clopen sets S_i such that $\mu(S_i) = 1$ for each i while $\bigcap_{i=1}^{\infty} S_i = \bigcap_{i=1}^{\infty} z(f_i) = \bigcap_{i=1}^{\infty} E_i = \varnothing$. But this contradicts the assumption that μ is σ–smooth on $\mathcal{S}$.

PROPOSITION 8. *μ is σ–smooth on $\mathcal{S}$ if and only if $Z(M^{\mu})$ is closed under the formation of denumerable intersections.*

Proof. We appeal to the fact that a denumerable intersection of zero (C_δ) sets is a zero (C_δ) set and the result of Lemma 4 for the proof.

We wish now to consider the extensions of the σ–smooth monotone 0–1 measures. We wish to show that these measures can be extended to the σ–algebra $\mathcal{B}$ generated by the zero sets. Henceforth we refer to $\mathcal{B}$ as the Baire sets.

PROPOSITION 9. *The following statements are equivalent:* (a) *μ is σ–smooth on $\mathcal{S}$;* (b) *$a(\mu)$ is a σ–algebra and $\hat{\mu}$ is σ–smooth on $a(\mu)$.*

Proof. Based on the results of Lemma 4 and Proposition 8 we need only show that if μ is σ–smooth on $\mathcal{S}$, then $a(\mu)$ is a σ–algebra. However, we observe that if $E_i \in a(\mu)$ and $CE_i \supset z(f_i)$ where $z(f_i) \in Z(M^{\mu})$ for each i, it follows from Proposition 8 that

$$\bigcap_{i=1}^{\infty} CE_i \supset \bigcap_{i=1}^{\infty} z(f_i) = z(f)$$

for some $f \in M^{\mu}$ and therefore $\bigcup_{i=1}^{\infty} E_i \in a(\mu)$. The other case $(E_i \supset z(f_i)$ where $f_i \in M^{\mu}$ for some $i)$ clearly leads to $\cup E_i \in a(\mu)$ and therefore $a(\mu)$ is a σ–algebra.

The measures $\hat{\mu}$ have the property that: $\hat{\mu}(E) =$ "sup" $\{\mu(z(f)) \mid z(f) \subset E\}$ for any set on which $\hat{\mu}$ is defined. Because of this, these measures will be referred to as *regular* measures.

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