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A Remark on Faithful Representations

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Matematica. — *A Remark on Faithful Representations.* Nota di MARTIN MOSKOWITZ, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Questa Nota vuole provare che in certi casi, data un'algebra di Lie $\mathfrak{G}$, esistono due gruppi di Lie, G e H, con algebra di Lie $\mathfrak{G}$, tali che G ha, ma H non ha mai, una rappresentazione lineare fedele.

The purpose of this Note is to prove, in various cases, a result which shows the futility of considering only linear Lie groups; namely that there always exist groups with the same Lie algebra which are non-linear. We denote by S the subgroup of the general linear group Gl $(n, \mathbf{C})$ consisting of matrices (g_{ij}) such that $g_{ij} = 0$ if $i < j$, and by the unipotent group U those elements of S for which each $g_{ii} = 1$. An easy calculation shows that the derived group S′ of S is contained in U. (They are actually equal.).

THEOREM. *Let* G *be a connected (non-abelian) nilpotent Lie group. Then there always exists a locally isomorphic group which has no continuous faithful finite dimensional linear representation (in* Gl $(n, \mathbf{C})$*).*

On the other hand, it should be borne in mind that by a Theorem of Cartan, for any nilpotent group G its simply connected covering group always has a continuous faithful finite dimensional linear representation (in fact, by unipotent matrices). See, for example G. Hochschild [5], Theorem (3.1), p. 219.

Proof. By considering the simply connected covering group $G^{\sim}$, we may evidently assume that G is simply connected. Let $\mathfrak{G}$ be its Lie algebra and $\mathfrak{G} \supset \mathfrak{G}_1 \supset \cdots \supset \mathfrak{G}_r = (0)$ be the descending central series. Then $\mathfrak{G}_{r-1}$ is non-zero and is contained in the center $\mathfrak{Z}(\mathfrak{G})$. Also $\mathfrak{G}_1 = [\mathfrak{G}\mathfrak{G}]$ the derived subalgebra. Since $\mathfrak{G}$ is non-abelian, $r > 1$ so that $[\mathfrak{G}, \mathfrak{G}] \cap \mathfrak{Z}(\mathfrak{G}) \neq (0)$. It follows that $G' \cap Z(G)_0 \neq (1)$ where $Z(G)_0$ (1) is the connected component of the identity of the center of G. Since G is simply connected, G′ is closed [5, Theorem (1.2), p. 135]. Also as a normal, even central, analytic subgroup of the simply connected nilpotent and therefore solvable group G, $G' \cap Z(G)_0$ is simply connected. Elementary abelian group theory tells us that $G' \cap Z(G)_0$ is a vector group (of positive dimension). Let Γ be any lattice in $G' \cap Z(G)_0$. Then Γ is a discrete central subgroup of G and therefore $G/\Gamma = H$ is a locally isomorphic group. In particular it is also nilpotent (and solvable). Suppose ρ is some continuous faithful finite dimensional representation of H. Since H is connected and solvable, it follows from the global version of Lie's Theorem

(*) Nella seduta del 13 maggio 1972.

(1) Since G is nilpotent Z (G) is actually connected.

(see [6], Sem. S. Lie exposé § 2) that there exists a basis of the representation space in which $\rho(H) \subseteq S$. If $\pi : G \rightarrow H$ denotes the canonical epimorphism, then $\pi(G') = H'$ and $\rho(H') = \rho(H)' \subseteq S' \subseteq U$. On the other hand, $\pi(G') \supseteq \pi(G' \cap Z(G)_0) = (G' \cap Z(G)_0)/\Gamma$ which is compact. Since ρ is continuous and faithful, it follows that U contains a non-trivial compact subgroup. Since the subgroup is non-trivial, n must be greater than 1. However, U is clearly diffeomorphic with $\mathbf{C}^{n(n-1)/2}$ and hence is simply connected. It is also evidently nilpotent and therefore solvable. As a simply connected solvable group, U has no non-trivial compact subgroups [5, Theorem (2.3), p. 138]. This contradiction proves the Theorem.

We remark that the case of the Theorem in which $G = N_3$, the Heisenberg group, is due to Birkoff via a very different method. (See [7], p. 191). Evidently our proof also goes through more generally if G is any solvable analytic group for which $\mathfrak{Z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ is non-zero. In particular, let G be any connected solvable linear algebraic group and let $\mathfrak{g} = \mathfrak{n} \oplus_\eta T$ be its semi-simple splitting where $\mathfrak{n}$ is the nilpotent radical and T is an abelian algebra of semi-simple automorphisms (see [1], p. 130). If $[\mathfrak{Z}(\mathfrak{n}) \cap [\mathfrak{n}, \mathfrak{n}], T] = 0$ then the above condition holds. For since $\mathfrak{n}$ is nilpotent, $\mathfrak{Z}(\mathfrak{n}) \cap [\mathfrak{n}\, \mathfrak{n}] \neq (0)$ as was shown above. If $u \neq 0$ is in the latter then our assumption implies $u \in \mathfrak{Z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$. For example, let $n \geq 3$ and $\mathfrak{g}$ be any subalgebra of the Lie algebra of all triangular $n \times n$ (real or complex) matrices $x = (x_{ij})$ such that $x_{11} = 0 = x_{nn}$ which contains $\mathfrak{n} = \{$triangular matrices such that $x_{ii} = 0$ for all $i\}$. Let T be the diagonal matrices of $\mathfrak{g}$. Then as is easily seen $\mathfrak{g}$ is the semi-direct sum of the ideal $\mathfrak{n}$ and the subalgebra T. Moreover, $\mathfrak{n}$ (which is the Lie algebra of U) has the property that $[\mathfrak{Z}(\mathfrak{n}), T] = (0)$ and $(0) \neq \mathfrak{Z}(\mathfrak{n}) \subseteq [\mathfrak{n}, \mathfrak{n}]$. Hence if G is the analytic subgroup of S (real or complex) corresponding to $\mathfrak{g}$ then G is a solvable linear group which has a locally isomorphic group without a faithful linear representation.

Our next result is of a somewhat different character.

THEOREM. *Let* G *be a connected and simply connected algebraic subgroup of* $\mathrm{Gl}(n, \mathbf{C})$ *defined over* $\mathbf{Q}$ *and* $G_{\mathbf{R}}$ *be the connected component of* 1 *of the real points of* G. *If* $G_{\mathbf{R}}$ *is not simply connected then although* $G_{\mathbf{R}}$ *evidently comes equipped with a faithful representation its simply connected covering group* $\tilde{G_{\mathbf{R}}}$ *has no faithful linear representation.*

For example, if $G = \mathrm{Sl}(n, \mathbf{C})$, $n \geq 2$, then G is connected and simply connected while $G_{\mathbf{R}} = \mathrm{Sl}(n, \mathbf{R})$. Here $\Pi_1(G_{\mathbf{R}}) = \mathbf{Z}$ if $n = 2$ and $\Pi_1(G_{\mathbf{R}}) = \mathbf{Z}_2$ if $n > 2$. Similarly if $G = \mathrm{S}p(n, \mathbf{C})$ for $n \geq 1$ then G is connected and simply connected. On the other hand $G_{\mathbf{R}}$ equals $\mathrm{S}p(n, \mathbf{R})$. Here $\Pi_1(G_{\mathbf{R}}) = \mathbf{Z}$ for $n \geq 1$. Thus $\mathrm{Sl}(n, \mathbf{R})^{\sim}$ for $n \geq 2$ and $\mathrm{S}p(n, \mathbf{R})^{\sim}$ for $n \geq 1$ have no faithful linear representations. Concerning the computation of the fundamental group of the various classical groups mentioned in this note see pages 342 and 345 of Helgeson [4].

Proof. We apply the first part of Theorem (3.3), p. 201 of [5] noting that this part of the Theorem makes no use of semi-simplicity. Let $\mathfrak{g}$ be the Lie algebra of $G_{\mathbf{R}}$. Then $\mathfrak{A}\oplus i\mathfrak{A}$, its complexification, is the Lie algebra of G. As in [5] let $\mathfrak{A} \xrightarrow{i} \mathfrak{A}\oplus i\mathfrak{A}$ be the canonical injection. Then there exists a real analytic homomorphism $\sigma : \tilde{G_{\mathbf{R}}} \to G$ with differential of $\sigma = i$. Then [5] tells us in this case that since $\tilde{G_{\mathbf{R}}}$ is simply connected that if ρ is any continuous finite dimensional representation of $\tilde{G_{\mathbf{R}}}$ then $\operatorname{Ker}\rho \supseteq \operatorname{Ker}\sigma$. But $\operatorname{Ker}\sigma = \Pi_1(G_{\mathbf{R}}) \neq (1)$. Thus $\tilde{G_{\mathbf{R}}}$ has no faithful representation.

COROLLARY. *If* G *is a connected and simply connected algebraic subgroup of* Gl $(n, \mathbf{C})$ *defined over* $\mathbf{Q}$ *and* $G_{\mathbf{R}}$ *is compact then* $G_{\mathbf{R}}$ *must also be simply connected.*

Proof. Suppose $G_{\mathbf{R}}$ is not simply connected. Then $\tilde{G_{\mathbf{R}}}$ has no faithful finite dimensional linear representations. Since $G_{\mathbf{R}}$ is compact it follows from [6] exposé § 22 that $\tilde{G_{\mathbf{R}}}$ is the direct sum of a vector group and a compact group. As such it has a faithful finite dimensional continuous (even unitary) representation. Here one takes the direct sum of such a representation of the compact part given by the Peter–Weyl Theorem with a representation of this type of the vector part in, for example, a torus of twice the dimension. Alternatively, one could simply apply Theorem (5.1), p. 31 of [3].

To see that the Corollary and Theorem above are not true if G is not simply connected (even in the semi-simple case) we consider the example G = the complex orthogonal group SO $(n, \mathbf{C})$, $n \geq 3$. Then $\Pi_1(G) = \mathbf{Z}_2$ but $G_{\mathbf{R}}$ is SO $(n, \mathbf{R})$ and is compact. Hence $G_{\mathbf{R}}$ is not simply connected and $\tilde{G_{\mathbf{R}}}$ (which is compact by Weyl's Theorem) has a faithful representation.

Finally, we remark that the results of the present Note complement in a natural way certain of the results of M. Goto in [2].

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