Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Nicolae Teleman

A characteristic ring of a Lie algebra extension

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **52** (1972), n.5, p. 708–711.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_52_5_708_0>

Topologia algebrica. — *A characteristic ring of a Lie algebra extension.* Nota II di NICOLAE TELEMAN, presentata (*) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — In questo lavoro costruisco un «anello caratteristico» per estensioni corte di algebre di Lie. In particolare mostro che ad ogni fibrato principale può associarsi una estensione di algebre di Lie in tal guisa che si ritrova così sostanzialmente la costruzione geometrica dell'anello caratteristico del fibrato dovuta a S. S. Chern e A. Weil.

Mostro inoltre, fra l'altro, che l'anello caratteristico qui considerato permette di introdurre un anello caratteristico per la coomologia di I. M. Gelfand e D. B. Fuks [4].

§ 6. APPLICATIONS

6.1. Let $E \xrightarrow{p} M$ be a smooth vector bundle over the manifold M ($\dim M = n$) with fiber $\mathbf{F}^N$, $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Let T denote the tangent bundle of M.

Then $\tilde{E} = \mathrm{ISO}(\mathbf{F}^N, E) \xrightarrow{\tilde{p}} M$ is the principal $GL(N, \mathbf{F})$ bundle associated with E.

There exists a natural right action of $GL(N, \mathbf{F})$ over $\tilde{E}$ induced by the composition

$$\mathrm{ISO}(\mathbf{F}^N, E) \cdot \mathrm{ISO}(\mathbf{F}^N, \mathbf{F}^N) \rightarrow \mathrm{ISO}(\mathbf{F}^N, E).$$

Let $\mathfrak{U} = (U_\alpha)_{\alpha \in \Lambda}$ be an open covering on M and $g_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow GL(N, \mathbf{F})$ a system of transition functions for the vector bundle E, and hence, at the same time, for the principal bundle $\tilde{E}$.

Let Y be a tangent field over E, invariant by the right. For any $\alpha \in \Lambda$, let be $Y_\alpha = Y\big|_{\tilde{p}^{-1}(U_\alpha)}$. The field Y_α can be uniquely described by the formula:

$$\begin{aligned} & Y_\alpha(x, g) = (X(x), A_\alpha(x) \cdot g) \quad \text{for} \\ (30) \qquad & (x, g) \in U_\alpha \times GL(N, \mathbf{F}), \\ & X_\alpha = \tilde{p}_* Y_\alpha, \ A_\alpha : U_\alpha \rightarrow gl(N, \mathbf{F}) = T_e GL(N, \mathbf{F}). \end{aligned}$$

Hence, locally, any right invariant tangent field Y over $\tilde{E}_\alpha = \tilde{p}^{-1}(U_\alpha)$ can be defined by a pair (X_α, A_α), where X_α is a tangent field over U_α and $A_\alpha : U_\alpha \rightarrow gl(N, \mathbf{F})$.

(*) Nella seduta dell'8 aprile 1972.

A direct calculus proves that over $\tilde{p}^{-1}(U_\alpha \cap U_\beta)$ the connection between the two descriptions (X_α, A_α), (X_β, A_β) of the same invariant field Y is:

$$(31) \qquad \begin{aligned} X_\alpha &= X_\beta \\ A_\beta &= (X_\beta g_{\beta\alpha}) g_{\beta\alpha}^{-1} + g_{\beta\alpha} A g_{\beta\alpha}^{-1} . \end{aligned}$$

If Y^1, Y^2 are two right invariant fields over $\tilde{E}$ and if (X^i, A^i_α), $i = 1, 2$, are their local descriptions, then

$$(32) \qquad [(X^1, A^1_\alpha), (X^2, A^2_\alpha)] = ([X^1, X^2], X^1 A^2_\alpha - X^2 A^1_\alpha - [A^1_\alpha, A^2_\alpha]).$$

PROPOSITION 13. *If $\tilde{E}$ is a principal bundle defined by the transition functions $g_{\beta\alpha} : U_\alpha \cap_\beta \rightarrow GL(N, \mathbf{F})$, then the set* P *of all right invariant tangent fields over* E *is in 1–1 correspondence with the sections in the real vector bundle $\check{E}$ with fibre in any point $x \in M : T_x M \oplus gl(n, \mathbf{F})$ and transition functions* (31).

P *is a real Lie–algebra whose local structure is described by* (32).

We observe that the projections

$$(33) \qquad (X_\alpha, A_\alpha) \xrightarrow{\pi_\alpha} X_\alpha$$

are compatible with the transition functions (31) and hence these define an epimorphism of vector bundles $\pi : \check{E} \rightarrow T$. The relations (32) show that

$$(34) \qquad \bar{\pi} : C^\infty(M, \check{E}) = P \rightarrow C^\infty(M, T)$$

is a Lie algebra homomorphism.

From (31), (33) we deduce that the vector bundle *ker* π is Hom (E, E) and hence we have the exact sequence of vector bundles (see [1])

$$(34) \qquad O \rightarrow \text{Hom}(E, E) \xrightarrow{\iota} \check{E} \rightarrow T \rightarrow 0 .$$

The upon considerations show that at the same time we have the exact sequence of Lie algebras

$$(35) \qquad \mathcal{E}(E) \equiv O \rightarrow C^\infty(M, \text{Hom}(E, E)) \rightarrow P \rightarrow C^\infty(M, T) \rightarrow 0 .$$

We refer now to the conditions (1)–(4) § 2. We consider two situations:

$$(36) \qquad R = \mathbf{R} \quad , \quad \mathcal{F} = C^\infty(M, \mathbf{R}) \text{ and } (\mathcal{E}) \equiv \mathcal{E}(E)$$

$$(37) \qquad R = \mathbf{R} \quad , \quad \mathcal{F} = \mathbf{R}, \quad \text{and } (\mathcal{E}) \equiv \mathcal{E}(E).$$

6.2. We analyse the case (36).

Let Γ be a linear connection for E. Let Ω_Γ denote the curvature tensor for the connection Γ.

It is known [5] that the connection Γ defines a parallel transport for the principal bundle $\tilde{E}$. It is easy to see that the last parallel transport is the parallel transport associated to a connection $\tilde{\Gamma}$ for the principal bundle $\tilde{E}$. But $\tilde{\Gamma}$ is not other than a $C^\infty(M, \mathbf{R})$–splitting in (35). In other words, Γ defines a connection $\tilde{\Gamma}$ for (35).

PROPOSITION 14.

$$\Omega_{\tilde{\Gamma}} = \Omega_\Gamma .$$

Proof. Immediate from the definitions.

PROPOSITION 15. *If the manifold* M *is connected, the algebra of constant invariant forms on* $H = C^\infty(M, \mathrm{Hom}(E, E))$ *coincides with the algebra of invariant forms in the sense of S.S. Chern and A. Weil* (see [2], § 2).

Proof. It is sufficient to prove the last assertion for any U_α, $\alpha \in \wedge$. Let φ be a constant invariant form over $H_\alpha = C^\infty(U_\alpha, \mathrm{Hom}(E|_{U_\alpha}, E|_{U_\alpha}))$. For any $X, e_1, \cdots, e_s \in C^\infty(U_\alpha, \mathrm{Hom}(E|_{U_\alpha}, E|_{U_\alpha}))$, we have by definition of φ and (32):

$$O = \sum_{i=1}^{s} \varphi(e_1, \cdots, [X, e_i], \cdots, e_s),$$

hence φ is, in any point $x \in U_\alpha$, the restriction at that point, of an invariant form in the sense of S. S. Chern and A. Weil.

We know $E_\alpha = E|_{U_\alpha}$ is trivial; hence Hom (E_α, E_α) is trivial too. If we consider now $X = (\bar{X}, O)$ over U_α (in the local description) we obtain:

$$X(\varphi(e_1, \cdots, e_s)) = \sum_{i=1}^{s} \varphi(e_1, \cdots, Xe_i, \cdots, e_s),$$

hence φ is a constant form.

THEOREM 15. *The characteristic ring of the vector bundle* E *and the characteristic ring of the associate extension* (35) *coincide.*

Proof. The bide cohomological rings have the same construction, with same elements.

6.3. We consider the second case, (37). The space $H = C^\infty(M, \mathrm{Hom}(E, E))$ admits a natural Frechet topology.

We consider only the *continuous* (with respect the Fréchet topology) " constant invariant " forms over H. If ∇ is a $C^\infty(M, \mathbf{R})$–splitting for $\mathcal{E}(E)$, then ∇ is also a $\mathbf{R}$–splitting for $\mathcal{E}(E)$. If φ is a continuous constant invariant form over H, then $\tilde{\varphi}(\Omega_\nabla, \cdots, \Omega_\nabla)$ is a closed form in the complex of I.M. Gelfand and D.B. Fuks [4]. Hence we obtain the

THEOREM 16. *If* E *is a real or complex vector bundle over* M, *and if we consider only the continuous constant invariant forms in the sense exposed upon, then we obtain for the conditions* (37) *a system of characteristic classes for* E *in the cohomological algebra of* I. M. Gelfand *and* D. B. Fuks [4].

6.4. Let E, F be two smooth **R** or **C** vector bundle (of the same dimension N) over M. Let $(g_{\beta\alpha})$, resp. $(h_{\beta\alpha})$ be a system of transition functions for E, resp. F. The bundle ISO (E, F) has as fibre the group GL $(N, \mathbf{R})$

or GL (N , **C**), but it does not admit, in general, a structure of principal bundle. We can consider over ISO (E , F) a particular class of tangent fields: fields which, locally, are a sum of a left-invariant tangent field and a right-invariant tangent field. We denote by (L — R) (E , F) the space of these fields. This space is closed by respect of the bracket and hence is a Lie algebra.

If we proceed as in 6.1., we obtain a vector bundle over M whose fiber is $\mathbf{R}^n \oplus (gl(N, \mathbf{F}) \oplus gl(N, \mathbf{F})/Z)$, $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, and Z is the centre of $gl(N, \mathbf{F})$. The transition functions of this vector bundle are:

$$(X, \mathbf{B} \# C) \mapsto (X, ((Xg_{\beta\alpha})g_{\beta\alpha}^{-1} + (\mathrm{Ad}\, g_{\beta\alpha})\, B) \# \\ \# (-(Xh_{\beta\alpha})\, h_{\beta\alpha}^{-1} + (\mathrm{Ad}\, h_{\beta\alpha})\, C) \tag{39}$$

where $\#$ denotes $\oplus/Z$.

We can associate a Lie algebra extension as in 6.1. and hence a characteristic ring (in the de Rham or Gelfand–Fuks cohomology).

We remark that the characteristic ring in the de Rham cohomology (in the complex case) is generally smaller than the characteristic ring generated by the characteristic rings of E, resp. F.

REFERENCES

[1] M. F. ATIYAH, *Complex analytic connections in fibre bundles*. « Trans. Amer. Math. Soc. », *85*, 181–207 (1957).

[2] R. BOTT and S. S. CHERN, *Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections*, « Acta Mathematica », *114*, 71 (1965).

[3] S. S. CHERN, *Complex manifolds*, Chicago 1956.

[4] I. M. GELFAND and D. B. FUKS, *The cohomology of the Lie algebra of smooth tangent fields*. « Funct. Analiz. Priloj. T3 », *3*, 32–52 (1969) (Russian).

[5] S. KOBAYASHI and K. NOMIZU, *Foundations of differential geometry*, vol. I, 1963, Interscience Publishers.

[6] Séminaire: *Sophus Lie*, I^e^ annèe: 1954–55, *Théorie des algébres de Lie, Topologie des groupes de Lie* Paris, 1955.