Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **52** (1972), n.5, p. 702–707.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_52_5_702_0>

Geometria differenziale. — *Relative associate curvature of a congruence and λ-pseudogeodesics of a Finsler subspace.* Nota di

CHANDRA MANI PRASAD, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Il presente lavoro tratta della estensione della prima curvatura relativa e della pseudogeodesica (Pan [2], [3]) di un sottospazio riemanniano rispetto a un sottospazio di Finsler.

1. INTRODUCTION

The metric function of a Finsler space F_n referred to local coordinates $x^i (i = 1, 2, \cdots, n)$ is denoted by $F(x, x')$. Let F_m with local coordinates $u^\alpha (\alpha = 1, 2, \cdots, m)$ be a subspace of F_n. Let $C: u^\alpha = u^\alpha(s)$ be a curve of F_m, s being its arc-length. At every point of C, we have a line-element denoted by (u^α, u'^α). The metric tensors $g_{\alpha\beta}(u, u')$ and $g_{ij}(x, x')$ of F_m and F_n respectively are related by

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x') B^i_\alpha B^j_\beta \tag{1.1}$$

where

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}, \quad u'^\alpha = du^\alpha/ds \quad \text{and} \quad x'^i = dx^i/ds. \tag{1.2}$$

The $n - m$ unit vectors $n^i_{(\sigma)}(x)$, $(\sigma = m+1, \cdots, n)$, normal to F_m, are given by the solutions of the equations [1]

$$n_{(\sigma)i} B^i_\alpha = g_{ij}(x, n_{(\sigma)}) n^j_{(\sigma)} B^i_\alpha = 0 \tag{1.3}$$

$$g_{ij}(x, n_{(\sigma)}) n^i_{(\sigma)} n^j_{(\nu)} = \cos(n_{(\sigma)}, n_{(\nu)}). \tag{1.4}$$

The covariant derivative of B^i_α with respect to u^β is given by

$$I^i_{\alpha\beta} = \sum_\nu B_{(\nu)\alpha\beta} n^i_{(\nu)} + w^i_{\alpha\beta} \tag{1.5}$$

where

$$w^i_{\alpha\beta} n_{(\nu)i} = 0, \quad I^i_{\alpha\beta} n_{(\nu)i} = \Omega_{(\nu)\alpha\beta}.$$

The quantities $\Omega_{(\nu)\alpha\beta}$ are called the second fundamental tensors of the subspace. The covariant derivative of $n^i_{(\sigma)}$ be given by

$$n^i_{(\sigma);\beta} = A^\delta_{(\sigma)\beta} B^i_\delta + \sum_\nu \nu^{(\nu)}_{(\sigma)\beta} n^i_{(\nu)} \tag{1.6}$$

where $A^\delta_{(\sigma)\beta}$ and $\nu^{(\nu)}_{(\sigma)\beta}$ are defined in [1].

(*) Nella seduta del 13 maggio 1972.

Let λ^i be a contravariant vector defining a congruence in F_n which at a point of F_m be given by the equations

$$\lambda^i(u) = t^\alpha(u)\,B^i_\alpha + \sum_\nu d_{(\nu)}\,n^i_{(\nu)}\,, \qquad (\nu = m+1,\cdots,n). \tag{1.7}$$

The components of the derived vector $\dfrac{\delta\lambda^i}{\delta s}$ are given by

$$\frac{\delta\lambda^i}{\delta s} = W^\alpha\,B^i_\alpha + \sum_\nu D_{(\nu)}\,n^i_{(\nu)} \tag{1.8}$$

where

$$W^\alpha = \delta t^\alpha/\delta s - \sum_\nu B_{(\nu)\beta\gamma}\,M^\alpha_{(\nu)}\,t^\gamma\,u'^\beta + \sum_\nu d_{(\nu)}\,A^\alpha_{(\nu)\beta}\,u'^\beta \tag{1.9}$$

and

$$D_{(\nu)} = B_{(\nu)\alpha\beta}\,t^\alpha\,u'^\beta + \frac{\delta d_{(\nu)}}{\delta s} + \Sigma d_{(\sigma)}\,\nu^{(\sigma)}_{(\nu)\beta}\,u'^\beta\,. \tag{1.10}$$

Consider $n-m$ congruences of curves given by linearly independent unit vectors $\mu^i_{(\sigma)}$ $(\sigma = m+1,\cdots,n)$ satisfying

$$\mu^i_{(\sigma)} = l^\alpha_{(\sigma)}\,B^i_\alpha + \sum_\nu C_{(\sigma\nu)}\,n^i_{(\nu)}\,. \tag{1.11}$$

Let the $n-m$ vectors $\mu^i_{(\sigma)}$ with m linearly independent vectors of F_m form a set of n linearly independent vectors of F_n. This is possible only when $|C_{(\sigma\nu)}| \neq 0$. These congruences are called μ–congruences.

2. RELATIVE ASSOCIATE CURVATURE OF A CONGRUENCE

Let us define, at each point of C, a set of $n-m$ unit vectors $N^i_{(\sigma)}$ $(\sigma = m+1,\cdots,n)$ with the following properties:

(*a*) The vector $N^i_{(\sigma)}$, for each σ, is a linear combination of $\mu^i_{(\sigma)}$ and dx^i/ds,

(*b*) every $N^i_{(\sigma)}$ is orthogonal with respect to dx^i/ds. We have, therefore,

$$\begin{cases} N^i_{(\sigma)} = a_{(\sigma)}\,\dfrac{dx^i}{ds} + b_{(\sigma)}\,\mu^i_{(\sigma)}\,, \\ g_{ij}(x\,,x')\,N^i_{(\sigma)}\,dx^j/ds = 0 \qquad \text{and} \\ g_{ij}(x\,,N_{(\sigma)})\,N^i_{(\sigma)}\,N^j_{(\sigma)} = 1\,. \end{cases} \tag{2.1}$$

The relations in (2.1) give

$$\begin{cases} a_{(\sigma)} = -\,b_{(\sigma)}\,\mu_{(\sigma)} \qquad \text{and} \\ b_{(\sigma)} = \pm\,\dfrac{(\bar\psi_{(\sigma)})^{1/2}}{(\varphi_{(\sigma)} - \mu^2_{(\sigma)})^{1/2}} \end{cases} \tag{2.2}$$

where

$$(2.3)\qquad \begin{cases} \mu_{(\sigma)} = g_{ij}(x, x')\, x'^i \mu^j_{(\sigma)} \\ \varphi_{(\sigma)} = g_{ij}(x, x')\, \mu^i_{(\sigma)} \mu^j_{(\sigma)} \\ \bar{\psi}_{(\sigma)} = g_{ij}(x, x')\, N^i_{(\sigma)} N^j_{(\sigma)} \end{cases}$$

(repeated index σ does not denote summation).

From the equations (1.11), (2.1) and (2.2), we get

$$(2.4)\qquad N^i_{(\sigma)} = \pm \frac{\left[\left(l^\alpha_{(\sigma)} - \mu_{(\sigma)} \frac{du^\alpha}{ds}\right) B^i_\alpha + \sum_\nu C_{(\sigma\nu)}\, n^i_{(\nu)}\right] (\bar{\psi}_{(\sigma)})^{1/2}}{(\varphi_{(\sigma)} - \mu^2_{(\sigma)})^{1/2}},$$

As the vectors $B^i_\alpha (\alpha = 1, 2, \cdots, m)$ and $n^i_{(\nu)} (\nu = m+1, \cdots, n)$ are linearly independent, $b_{(\nu)} \neq 0$ and $|C_{(\sigma\nu)}| \neq 0$. It may be shown that $N^i_{(\sigma)}$ and B^i_α are linearly independent. Also the vectors $N^i_{(\sigma)}$ are linearly independent.

Eliminating $n^i_{(\nu)}$ with the help of (1.8) and (2.4), we have

$$(2.5)\qquad \frac{\delta\lambda^i}{\delta s} = W^\alpha B^i_\alpha + \sum_\nu D_{(\nu)} \Big[\pm N^i_{(\sigma)} (\bar{\psi}_{(\sigma)})^{-1/2} (\varphi_{(\sigma)} - \mu^2_{(\sigma)})^{1/2} - \\ - \left(l^\alpha_{(\sigma)} - \mu_{(\sigma)} \frac{du^\alpha}{ds}\right) B^i_\alpha\Big] \bar{C}_{(\sigma\nu)}$$

where

$$\bar{C}_{(\sigma\nu)} = \frac{\text{cofactor of } C_{(\sigma\nu)} \text{ in } |C_{(\sigma\nu)}|}{|C_{(\sigma\nu)}|}.$$

Simplifying (2.5), we obtain

$$(2.6)\qquad \frac{\delta\lambda^i}{\delta s} = \nu^\alpha B^i_\alpha \pm \sum_\sigma \sum_\nu D_{(\nu)} \bar{C}_{(\sigma\nu)} (\varphi_{(\sigma)} - \mu^2_{(\sigma)})^{1/2} (\bar{\psi}_{(\sigma)})^{-1/2} N^i_{(\sigma)}$$

where

$$(2.7)\qquad \nu^\alpha = W^\alpha - \sum_\sigma \sum_\nu D_{(\nu)} \left(l^\alpha_{(\sigma)} - \mu_{(\sigma)} \frac{du^\alpha}{ds}\right) \bar{C}_{(\sigma\nu)}.$$

DEFINITION (2.1). The vector ν^α given by the equation (2.7) is called the *relative associate curvature vector* of the congruence λ^i (with respect to the μ–congruences) in the direction of the curve C. The scalar νK defined by

$$(2.8)\qquad \nu K^2 = g_{\alpha\beta}(u, u')\, \nu^\alpha \nu^\beta$$

is called the relative associate curvature of the congruence λ^i.

DEFINITION (2.2). The vector W^α is called the associate curvature vector of λ^i and $W^\alpha = 0$ is called the λ–geodesic [4].

THEOREM (2.1). *The relative associate curvature vector of a congruence λ^i (in the direction of a curve* C*) is equal to the vector W^α if the derived vector $\delta\lambda^i/\delta s$ of the congruence λ^i is tangential to the subspace.*

Proof: The proof of the Theorem follows from the equations (1.8) and (2.7).

THEOREM (2.2). *The necessary and sufficient condition that the derived vector* $\delta\lambda^i/\delta s$ *of* λ^i *be equal to its relative associate curvature vector* $\nu^\alpha B^i_\alpha$ *(both the vectors being considered along the curve* C) *is that the first curvature vector of* λ^i *(with respect to* F_n) *does not lie in a space spanned by the normals.*

Proof: If $\delta\lambda^i/\delta s$ is not normal to the subspace, $D_{(\nu)} = 0$ and from (2.6), we get

$$\delta\lambda^i/\delta s = \nu^\alpha B^i_\alpha . \tag{2.9}$$

Conversely, let (2.9) hold. Since $N^i_{(\sigma)}$ are linearly independent, we get from (2.6),

$$\sum_\nu \bar{C}_{(\sigma\nu)} D_{(\nu)} (\bar{\psi}_{(\sigma)})^{-1/2} (\varphi_{(\sigma)} - \mu^2_{(\sigma)})^{1/2} = 0 \qquad \text{for} \quad \sigma = m+1, \cdots, n. \tag{2.10}$$

From (2.2), we have

$$\pm (\bar{\psi}_{(\sigma)})^{-1/2} (\varphi_{(\sigma)} - \mu^2_{(\sigma)})^{1/2} = \frac{1}{b_{(\sigma)}} \neq 0 ,$$

so that the equation (2.10) reduces to

$$\sum_\nu \bar{C}_{(\sigma\nu)} D_{(\nu)} = 0 , \qquad \text{for} \quad \nu = m+1, \cdots, n \tag{2.11}$$

and since $\bar{C}_{(\sigma\nu)} \neq 0$, we get

$$D_{(\nu)} = 0 , \qquad \text{for} \quad \nu = m+1, \cdots, n.$$

This completes the proof.

From the Theorems (2.1) and (2.2), we have

COROLLARY (2.1). *A necessary and sufficient condition that the three vectors* $\delta\lambda^i/\delta s$, $W^\alpha B^i_\alpha$ *and* $\nu^\alpha B^i_\alpha$ *be equal is that the vector* $\delta\lambda^i/\delta s$ *does not lie in a space spanned by the normals.*

In particular, if the congruence λ^i is tangential to the subspace, $d_{(\nu)} = 0$, but $t^\alpha \neq 0$ and then $\lambda^i = B^i_\alpha t^\alpha$. The equations (2.6) and (2.7) now respectively take the form

$$\frac{\delta\lambda^i}{\delta s} = \nu^\alpha B^i_\alpha \pm \sum_\sigma \sum_\nu \bar{C}_{(\sigma\nu)} B_{(\nu)\beta\gamma} t^\beta \frac{du^\gamma}{ds} (\bar{\psi}_{(\sigma)})^{-1/2} (\varphi_{(\sigma)} - \mu^2_{(\sigma)})^{1/2} N^i_{(\sigma)} \tag{2.12}$$

and

$$\nu^\alpha = \delta t^\alpha/\delta s + \sum_\sigma \sum_\nu B_{(\nu)\beta\gamma} t^\beta \frac{du^\gamma}{ds} \left\{ \delta_{(\sigma\nu)} M^\alpha_{(\sigma)} - \bar{C}_{(\sigma\nu)} \left(l^\alpha_{(\sigma)} - \mu_{(\sigma)} \frac{du^\alpha}{ds} \right) \right\} . \tag{2.13}$$

If we write

$$\delta t^\alpha/\delta s = dt^\alpha/ds + \Gamma^{*\alpha}_{\beta\gamma} t^\beta \frac{du^\gamma}{ds} , \tag{2.14}$$

the equations (2.13) can be written as

$$\nu^{\alpha} = \mathrm{d}t^{\alpha}/\mathrm{d}s + \mathrm{R}^{\alpha}_{\beta\gamma}\, t^{\beta}\, u'^{\gamma} \tag{2.15}$$

where

$$\mathrm{R}^{\alpha}_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma} + \sum_{\sigma}\sum_{\nu} \mathrm{B}_{(\nu)\beta\gamma} \left\{ \delta_{(\sigma\nu)} \mathrm{M}^{\alpha}_{(\sigma)} - \bar{\mathrm{C}}_{(\sigma\nu)} \left(l^{\alpha}_{(\sigma)} - \mu_{(\sigma)} \frac{\mathrm{d}u^{\alpha}}{\mathrm{d}s} \right) \right\}.$$

The quantities $\mathrm{R}^{\alpha}_{\beta\gamma}$ are called the relative connection parameters of the subspace [5].

Moreover, in this particular case, the Theorems (2.1) and (2.2) run as follows:

THEOREM (2.1 a). *A sufficient condition that the relative associate curvature vector* ν^{α} *of the vector-field* λ^{i} *(tangential to* F_m*) along* C *be equal to its associate curvature vector* $\delta t^{\alpha}/\delta s$ *is that the vector-field* t^{α} *is conjugate with respect to* C.

THEOREM (2.2 a). *A necessary and sufficient condition that the derived vector* $\delta\lambda^{i}/\delta s$ *of the tangential vector-field* λ^{i} *be equal to the relative associate curvature vector* $\nu^{\alpha}\mathrm{B}^{i}_{\alpha}$ *(both the vectors are considered along the same curve* C*) is that the vector-field* t^{α} *is conjugate with respect to* C.

3. λ–PSEUDOGEODESICS

A curve of the subspace is said to be a λ–psuedogeodesic (with respect to μ–congruences) if at each of its points the relative associate curvature of the congruence λ^{i} vanishes identically.

For such a λ–pseudogeodesic, we have $\nu^{\alpha} = 0$. Its equation, therefore, is given by

$$\mathrm{W}^{\alpha} - \sum_{\sigma}\sum_{\nu} \mathrm{D}_{(\nu)} \left(l^{\alpha}_{(\sigma)} - \mu_{(\sigma)} \frac{\mathrm{d}u^{\alpha}}{\mathrm{d}s} \right) \bar{\mathrm{C}}_{(\sigma\nu)} = 0\,. \tag{3.1}$$

THEOREM (3.1). *A λ–pseudogeodesic relative to the μ–congruences is a λ–geodesic if the first curvature vector of* λ^{i} *(with respect to* F_n*) does not lie in a variety spanned by the normals.*

Proof: From the equation (3.1), the definition (2.2) and the condition $\mathrm{D}_{(\nu)} = 0$, the proof of the Theorem follows.

If, in particular, $\lambda^{i} = x'^{i}$, the equation (3.1) then takes the form

$$p^{\alpha} + \sum_{\sigma}\sum_{\nu} \mathrm{B}_{(\nu)\beta\gamma} \frac{\mathrm{d}u^{\beta}}{\mathrm{d}s} \frac{\mathrm{d}u^{\gamma}}{\mathrm{d}s} \left\{ \delta_{(\sigma\nu)} \mathrm{M}^{\alpha}_{(\sigma)} - \bar{\mathrm{C}}_{(\sigma\nu)} \left(l^{\alpha}_{(\sigma)} - \mu_{(\sigma)} \frac{\mathrm{d}u^{\alpha}}{\mathrm{d}s} \right) \right\} = 0 \tag{3.2}$$

which is the equation of the pseudogeodesic of the subspace. In a hypersurface, the equation (3.2) reduces to the equation of a union curve (relative to μ–congruence) [5]. Hence *the pseudogeodesics of the hypersurface are the union curves.*

When $\lambda^i = x'^i$, the λ–pseudogeodesics are the pseudogeodesics which, in view of the equation (2.15), are given by the equations

$$p^\alpha + R^\alpha_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = 0 . \tag{3.3}$$

As the geodesics of the subspace with the equations

$$p^\alpha + \Gamma^{*\alpha}_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = 0$$

are the auto-parallel curves, the *pseudogeodesics are considered as the auto-relative parallel curves of the subspace.* But it may be remarked that the λ–pseudogeodesics are not auto-relative parallel curves of the Finsler subspace.

The Author is grateful to Prof. K. B. Lal for his help in the preparation of this paper.

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