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Remarks on fixed points

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Analisi funzionale. — *Remarks on fixed points.* Nota di SIMEON REICH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si fanno alcune osservazioni sui Teoremi relativi ai punti fissi e si stabiliscono nuove proprietà sulle trasformazioni non espansive in certi spazi di Banach.

0. INTRODUCTION

This Note contains miscellaneous remarks on fixed points in metric and Banach spaces. Emphasis is laid on non-expansive mappings. The main results are Theorems 1.3, 2.1 and 4.3.

1. KANNAN'S FUNCTIONS

Kannan [17, Theorem 2] has established the following result.

THEOREM 1.A. *Let* (M , d) *be a compact metric space and let a continuous* $T : M \to M$ *satisfy*

$$d(Tx , Ty) \leq \frac{1}{2} [d(x , Tx) + d(y , Ty)] \tag{1.1}$$

for all x *and* y *in* M. *Suppose that*

(1.2) $d(x , Tx)$ *is not constant on any closed subset of* M *which contains more than one point and is invariant under* T.

Then T *has a fixed point.*

In view of Kannan's fixed point Theorem [15, p. 73; 16, p. 406], it is somewhat surprising that T is assumed to be continuous. Indeed this assumption is redundant.

PROPOSITION 1.1. *Theorem* 1.A *remains valid when the continuity requirement imposed on* T *is dropped.*

Proof. Let $X \subset M$ be minimal with respect to being non-empty, closed and invariant under T. If X contains more than one point, there are w and z in X with $r = d(w , Tw) < d(z , Tz)$. Put $A = \{x \in X : d(x , Tx) \leq r\}$ and let Y be the closure of T(A). If $y \in Y$, then y is the limit of a sequence $\{Tx_n\}$ with $d(x_n , Tx_n) \leq r$ for each n. Since $d(y , Ty) \leq d(y , Tx_n) + \frac{1}{2} d(x_n , Tx_n) + \frac{1}{2} d(y , Ty)$ it follows that $d(y , Ty) \leq r$. Thus $Y \subset A$ and $T(Y) \subset Y$. This is a contradiction because Y is a proper subset of X.

(*) Nella seduta del 13 maggio 1972.

Condition (1.2) cannot be omitted [30, p. 10], but we can get rid of (1.2) by sharpening (1.1).

PROPOSITION 1.2. *Let* (M, d) *be a compact metric space and let* $T: M \to M$. *If* T *satisfies*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)] \tag{1.3}$$

for all $x, y \in M$ *with* $Tx \neq Ty$, *then it has a fixed point.*

Proof. Let $m = \inf\{d(x, Tx) : x \in M\}$. Choose a sequence $\{x_n\}$ such that $d(x_n, Tx_n) \to m$. A subsequence $\{y_k\} \subset \{Tx_n\}$ converges to a certain $y \in M$. Since $d(y, Ty) \leq d(y, y_k) + d(Tx_k, Ty) \leq d(y, y_k) + \frac{1}{2} d(x_k, Tx_k) + \frac{1}{2} d(y, Ty)$, we obtain $d(y, Ty) \leq 2\, d(y, y_k) + d(x_k, Tx_k)$. Thus $m = d(y, Ty)$. Were m positive, the contradiction $d(Ty, T^2 y) < d(y, Ty)$ would occur.

Sometimes (1.1) suffices to guarantee the existence of a fixed point.

Recall that a non-empty bounded convex subset K of a normed linear space E is said to have normal structure if for each convex subset S of K which contains more than one point, there is a point $x \in S$ which is non-diametral (that is, $\sup\{\|x - y\| : y \in S\}$ is strictly less than the diameter of S).

Although the argument presented in [31, Section 1.iii] can be adapted to yield a proof of our next result, we prefer to use a variant of an idea which appears in [4, p. 290] and [12, p. 1206].

THEOREM 1.3. *Let* C, *a non-empty weakly compact convex subset of a normed linear space, possess normal structure. If* $T: C \to C$ *satisfies*

$$\|Tx - Ty\| \leq \frac{1}{2}[\|x - Tx\| + \|y - Ty\|] \tag{1.4}$$

for all x *and* y *in* C, *then it has a fixed point.*

Proof. Let $t(x) = \|x - Tx\|$, $q = \inf\{t(x) : x \in C\}$, $r > q$ and $A = \{x \in C : t(x) \leq r\}$. If y belongs to the convex hull of $T(A)$ and $z \in C$, then $\|z - Tz\| \leq 2\|z - y\| + r$. It follows that the closed convex hull of $T(A)$ is a subset of A. Hence $Q = \{x \in C : t(x) = m\}$ is not void. Let K be the convex hull of $T(Q)$. K is contained in Q and invariant under T. Each of its points is diametral (cf. [17, Theorem 5]). Thus K is a singleton. This completes the proof.

This result answers completely a question we raised in [30, p. 11] (a partial answer appears in [31, Section 1.iii]). It shows that the continuity assumption in [18, Theorem 3] is superfluous.

2. ACCRETIVE OPERATORS

Let Q be a non-empty subset of a real Banach space E. We shall denote the boundary of Q by bdy(Q), its interior by int(Q), its closure by cl(Q) and its convex hull by co(Q). bdg(Q) will stand for the set of bounding points of Q, that is $\{x \in Q : x$ does not belong to the core of $Q\}$. If $x \in Q$ we put (after Halpern [13, p. 87]) $I_Q(x) = \{z \in E : z = x + c(y - x)$ for some $y \in Q$ and $c \geq 0\}$. If $x \notin \mathrm{bdg}(Q)$, then $I_Q(x) = E$. If Q is convex with non-empty interior we choose for each $y \in \mathrm{bdy}(Q)$ a support functional f_y. This functional satisfies $\langle x, f_y\rangle \leq \langle y, f_y\rangle$ and $\langle z, f_y\rangle < \langle y, f_y\rangle$ where $x \in Q$ and $z \in \mathrm{int}(Q)$.

Let C be a non-empty closed convex subset of E. We shall be interested in three conditions which may be satisfied by a mapping $T : C \to E$.

(2.1) $Ty \in I_C(y)$ for each $y \in \mathrm{bdg}(C)$;

(2.2) $\langle Ty, f_y\rangle \leq \langle y, f_y\rangle$ for each $y \in \mathrm{bdy}(C)$;

(2.3) For some $w \in \mathrm{int}(C)$ $Ty - w \neq m(y - w)$ for all $y \in \mathrm{bdy}(C)$ and $m > 1$.

In the last two conditions we assume of course that $\mathrm{int}(C) \neq \varnothing$. $(2.1) \Rightarrow (2.2) \Rightarrow (2.3)$, but the implications in the other direction do not hold in general.

A function $A : C \to E$ is said to be accretive (see for example [19, p. 141]) if for each positive r

$$\|x + rAx - y - rAy\| \geq \|x - y\| \tag{2.4}$$

for all x and y in C. It will be called strongly accretive if for each positive r and $x \in C$ there is a number $k(x, r) < 1$ such that

$$k(x, r)\,\|x + rAx - y - rAy\| \geq \|x - y\| \tag{2.5}$$

for each $y \in C$. A mapping $T : C \to E$ is said to be non-expansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{2.6}$$

for all $x, y \in C$. It is called (after Kirk [21, p. 567]) a generalized contraction if for each $x \in C$ there is a number $\alpha(x) < 1$ such that

$$\|Tx - Ty\| \leq \alpha(x)\,\|x - y\| \tag{2.7}$$

for all $y \in C$. Let I denote the identity function. If T is non-expansive (a generalized contraction), then $I - T$ is accretive (strongly accretive). If for some $k < 1$ T satisfies

$$\|Tx - Ty\| \leq k\,\|x - y\| \tag{2.8}$$

for all x and y in C, it is called a strict contraction.

THEOREM 2.1. *Let* C *be a non-empty bounded closed convex subset of a real Banach space* E. *Let a Lipschitzian* $T : C \to E$ *satisfy either* (2.1) *or* (2.2). *If* $I - T$ *is accretive* (*strongly accretive*) *and* C *has the fixed point property for non-expansive functions* (*generalized contractions*), *then* T *has a fixed point.*

Proof. Choose a positive r so that tT may be a strict contraction where $t = r/(r+1)$. $B = [I + r(I - T)]^{-1}$ is single-valued and non-expansive (a generalized contraction) on its domain D. Let $z \in C$ and $S = tT + (1-t)z$. If $v = x + c(y - x)$ where x and y are in C and $c \geq 0$, then $tv + (1-t)z = x + b(u - x)$ where $b = 1 + tc - t$ and $bu = tcy + (1-t)z$. If $y \in \mathrm{bdy}(C)$ and $\langle v, f_y \rangle \leq \langle y, f_y \rangle$, then $\langle tv + (1-t)z, f_y \rangle \leq t\langle y, f_y \rangle + (1-t)\langle y, f_y \rangle = \langle y, f_y \rangle$. Thus $S : C \to E$ satisfies (2.1) or (2.2), as the case may be. S has a fixed point by [33, Section 2] (or by [32, Theorems 3.3 and 3.5]). It follows that $C \subset D$. The fixed point of $B : C \to C$ is also a fixed point for T.

COROLLARY 2.2. *Let* C, *a non-empty weakly (weak star) compact convex subset of a real (conjugate) Banach space* E, *possess normal structure. If a non-expansive* $T : C \to E$ *satisfies either* (2.1) *or* (2.2), *then it has a fixed point.*

COROLLARY 2.3. *Let* C *be a non-empty weakly (weak star) compact convex subset of a real (conjugate) Banach space* E. *If a generalized contraction* $T : C \to E$ *satisfies either* (2.1) *or* (2.2), *then it has a fixed point.*

Those parts of Corollaries 2.2 and 2.3 which deal with condition (2.1) were established by a different, more direct method in [34]. The weak (weak star) lower-semicontinuity of the (conjugate) norm turned out to be useful there.

Theorem 2.1 and its corollaries improve upon recent results of Kirk [20, p. 821], [22, pp. 91 and 92] and Assad–Kirk [1, Theorem 3]. They assume that the image of bdy (C) under T is contained in C.

Sometimes we can replace (2.2) by (2.3). Here is a simple example. Recall that a function $V : C \to E$ is said to be strongly compact if $x_n \rightharpoonup x$ in C implies $Vx_n \to Vx$.

PROPOSITION 2.4. *Let* C *be a non-empty bounded closed convex subset of a Hilbert space* H. *Let* $T : H \to H$ *satisfy either* (2.1) *or* (2.3) *on* C. *If* $T = U + V$ *where* U *is Lipschitzian,* $I - U$ *is accretive and* V *is strongly compact, then* T *has a fixed point.*

We omit the details.

This Proposition improves upon a result due to Kluge [23, p. 321], who assumed that $U(x) + V(y)$ belongs to C for all x and y in C.

It might be of interest to determine whether Theorem 2.1 remains valid when (2.2) is replaced by (2.3).

We take this opportunity to remark that the following two results are true. They can be inferred from [33, Section 2] and [14, Theorem 4.1] respectively.

PROPOSITION 2.5. *Let* C *be a non-empty closed convex subset of a Banach space* E. *If a strict contraction* $T : C \to E$ *satisfies* (2.1), *then it has a fixed point.*

PROPOSITION 2.6. *Let* C *be a non-empty closed convex bounded subset of a strictly convex reflexive Banach space* E. *If an isometry* $T : C \to E$ *satisfies*

$$(2.9) \qquad Ty \in \mathrm{cl}(I_C(y)) \quad \textit{for each} \quad y \in \mathrm{bdg}(C),$$

then it has a fixed point.

Proposition 2.6 is an extension of [7, Theorem 7.1] which in turn improves upon the result in [6].

An elegant proof of [14, Theorem 4.1] can be obtained by adapting Fan's arguments in [8, p. 309] and [9, p. 235].

3. SET-VALUED FUNCTIONS

As an example of a possible fixed point theorem for set-valued non-expansive functions, we present an improvement of [32, Theorem 5.4].

Let (CC (E) , H) denote the space of all non-empty compact convex subsets of a Banach space E endowed with the Hausdorff metric H. Let $S \subset E$ be non-empty. A continuous $F : S \to CC(E)$ is said to be compact if $F(B)$ is contained in a compact subset of E for each bounded subset B of S, non-expansive if $H(F(x), F(y)) \leq \|x - y\|$ for all $x, y \in S$, strongly compact if $x_n \rightharpoonup x$ in S implies $F(x_n) \to F(x)$ in (CC (E) , H). If $U : S \times S \to \to CC(E)$, then the function $F : S \to CC(E)$ defined by $F(x) = U(x, x)$ for all $x \in S$ will be called the function associated with U.

A Banach space E is said to satisfy Opial's condition [27, p. 592] if $x_n \rightharpoonup x$ in E implies that $\liminf \|x_n - y\| > \liminf \|x_n - x\|$ for all $y \neq x$. A Hilbert space satisfies this condition.

PROPOSITION 3.1. *Let* C *be a non-empty weakly compact convex subset of a Banach space* E *which satisfies Opial's condition, and let* $F : C \to E$ *be associated with some* $U : E \times C \to CC(E)$ *where for a fixed* $y \in C$ $U(\cdot, y)$ *is non-expansive on* E, *and for a fixed* $x \in E$ $U(x, \cdot)$ *is strongly compact. If* F *satisfies either*

(3.1) $$F(y) \cap I_C(y) \neq \varnothing \quad \textit{for each} \quad y \in \text{bdg}(C),$$

or

(3.2) *For some* $w \in \text{int}(C)$ $z - w \neq m(y - w)$ for all $y \in \text{bdy}(C)$, $z \in F(y)$ *and* $m > 1$,

then it has a fixed point.

Proof. F enjoys the following property:

(3.3) If $\{x_n\} \subset C$, $x_n \rightharpoonup x$ and $x_n - w_n \to z$ where $w_n \in F(x_n)$, then $z \in x - F(x)$.

Indeed, if the assumptions of (3.3) hold, then for each n we can find $y_n \in U(x, x_n)$ such that $\|w_n - y_n\| \leq \|x_n - x\|$. Since $U(x, \cdot)$ is compact and strongly compact on C, we may assume that $y_n \to y$ for some $y \in U(x, x)$. Now $\|x_n - z - y\| \leq \|x_n - w_n - z\| + \|y_n - y\| + \|x_n - x\|$, so that $x = z + y$. An appeal to the results of [32] completes the proof.

If E is a Hilbert space $b_C = b_E$ (cf. [26, p. 932]) and U may be assumed to be defined only on $C \times C$. Here $b_M(B) = \inf\{r > 0 : B$ can be covered by a finite number of balls with centers in M and radius $r\}$ where $B \subset M \subset E$ and B is bounded, is the "ball measure of non-compactness".

This result extends the Theorems of Markin [25, p. 639] and Lami Dozo [24, p. 44]. It is also a partial extension of the Theorems in [24, p. 27] and [1, Theorems 2 and 4].

4. Iterations

Let E be a real Banach space, E^* its dual and $J : E \to E^*$ a normalized duality mapping [5, p. 50]. The norm of E is uniformly Gâteaux differentiable if and only if E^* is weak star uniformly rotund if and only if J is unique and uniformly continuous on bounded subsets of E from the strong topology of E to the weak star topology of E^* (cf. [2, p. 90] and [3, p. 303]).

PROPOSITION 4.1. *Let* E *be a real Banach space*, $T : E \to E$ *a non-expansive mapping and* $S = I - T$. *If* E^* *is weak star uniformly rotund, then* $\mathrm{cl}(S(E))$ *is convex.*

Proof. We need only show that $\mathrm{co}(S(E)) \subset \mathrm{cl}(S(E))$. Let $x \in \mathrm{co}(S(E))$. Then $x = \sum_{i=1}^{n} p_i y_i$ where $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$ and $y_i \in S(E)$ for each i. For each positive t there are unique points x_t and $y_{i,t}$ in E which satisfy $x = Sx_t + tx_t$ and $y_i = Sy_{i,t} + ty_{i,t}$. $\langle Sy_{i,t} - Sx_t, J(y_{i,t} - x_t)\rangle \geq 0$, because T is non-expansive. This leads to $t^2 \sum_{i=1}^{n} p_i \|x_t - y_{i,t}\|^2 \leq \sum_{i=1}^{n} p_i \langle x - y_i, J(tx_t - ty_{i,t}) - J(tx_t)\rangle$. The last inequality, combined with the uniform continuity of J, implies that $tx_t \to 0$ as $t \to 0$ because $\{y_{i,t} : t > 0\}$ is bounded for each i. This completes the proof.

In case E is a Hilbert space, or more generally a Banach space with a uniformly rotund dual, this result is due to Pazy [28, p. 238], [29]. In fact, it is possible to prove Proposition 4.1 by adapting Pazy's arguments in [28]. The proof presented above was inspired by Gossez [11, p. 77].

Uniformly rotund Banach spaces satisfy the following condition (cf. [10, p. 555]):

(4.1) For any convex set K, every sequence $\{x_n\}$ in K satisfying

$$\lim \|x_n\| = \inf\{\|x\| : x \in K\} \quad \text{converges.}$$

Let Q be a non-empty closed subset of a Banach space E which satisfies (4.1). $\mathrm{cl}(\mathrm{co}(Q))$ contains a unique element of least norm. We shall say (after Pazy [28, p. 237]) that Q has the minimum property if this element belongs to Q.

Let N denote the set of all non-negative integers. Let $\{c_n : n \in N\}$ be a sequence of real numbers which satisfy

$$(4.2) \qquad 0 < c_n \leq 1 \quad \text{for all} \quad n \in N ;$$

$$(4.3) \qquad \sum_{i=0}^{\infty} c_i \quad \text{diverges.}$$

The following proposition is Theorem 2 of [35].

PROPOSITION 4.2. *Let* $x_0 \in C$, *a closed convex subset of a Banach space* E *which satisfies* (4.1), let $T : C \to C$ *be non-expansive and let the sequence* $\{x_n : n \in N\}$ *be defined by*

$$(4.4) \qquad x_{n+1} = (1 - c_n)\, x_n + c_n T x_n \quad , \quad n \in N.$$

If cl(S(C)) *has the minimum property, then* $x_{n+1}/a_n \to -v$ *where* v *is the element of least norm in* cl(S(C)) *and* $a_n = \sum_{i=0}^{n} c_i$, $n \in N$.

Combining the last two Propositions we obtain an improvement of [28, Corollary 2].

THEOREM 4.3. *Let* $x_0 \in E$, *a Banach space whose norm is uniformly Gâteaux differentiable*, let $T : E \to E$ *be non-expansive and let* $\{x_n\}$ *be defined by* (4.4). *If* E *satisfies* (4.1), *then* $x_{n+1}/a_n \to -v$ *where* v *is the element of least norm in* cl(S(E)).

If, in addition, E is uniformly rotund, it follows that

(i) $0 \in S(E)$ if and only if $\{x_n\}$ is bounded for every $x_0 \in E$ and every sequence $\{c_n\}$;

(ii) $0 \notin \mathrm{cl}(S(E))$ if and only if $\lim \|x_{n+1}\|/a_n > 0$ for every $x_0 \in E$ and every sequence $\{c_n\}$;

(iii) $0 \in \mathrm{cl}(S(E))$, but $0 \notin S(E)$ if and only if $\{x_n\}$ is unbounded, but $x_{n+1}/a_n \to 0$ for every $x_0 \in E$ and every sequence $\{c_n\}$.

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