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A Characterization of Smooth α-Lipschitz Mappings on a Hilbert Space

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Analisi funzionale. — *A Characterization of Smooth α–Lipschitz Mappings on a Hilbert Space* (*). Nota di George R. Sell, presentata (**) dal Socio G. Sansone.

Riassunto. — Un esempio di trasformazione α–Lipschitziana è dato da $T = A + B$ dove A è compatta e B è Lipschitziano. In questo articolo si mostra che se T è una trasformazione α–Lipschitziana regolare su uno spazio di Hilbert, allora vale anche il viceversa. Inoltre, ciò che è più importante, mostriamo che il coefficiente di Lipschitz per B può essere scelto arbitrariamente prossimo all' α–modulo di T.

I. Introduction and Statement of Main Result

In his book [5], K. Kuratowski introduced the concept of the α–measure of a bounded set S in a metric space X. Precisely, $\alpha(S)$ is defined to be the infimum over all $\varepsilon > 0$ such that S can be covered by a finite number of sets, each of diameter less than or equal to ε. A mapping $T : X \to X$ is said to be an α–Lipschitz mapping if T maps bounded sets into bounded sets and if there is a real number k, $0 \leq k < \infty$, such that

$$\alpha(T(S)) \leq k\alpha(S) \tag{1}$$

for all bounded sets $S \subset X$. The infimum over all k that satisfy (1) is said to be the α–module of T, and we shall denote this by α_T.

If T is a compact operator, then it is α–Lipschitz and $\alpha_T = 0$. If T is a Lipschitz continuous operator, then it is α–Lipschitz and $\alpha_T \leq k_T$, where k_T denotes the Lipschitz coefficient of T, that is, k_T is the infimum over all k that satisfy

$$\rho(Tx, Ty) \leq k\rho(x, y)$$

for all $x, y \in X$. (Here we use ρ to denote the metric on X). If X is also a linear space, then the sum of two α–Lipschitz mappings $T = R + S$ is α–Lipschitz, and $\alpha_T \leq \alpha_R + \alpha_S$. In particular, if $T = R + S$ where R is compact and S is Lipschitz continuous, then T is α–Lipschitz and $\alpha_T = \alpha_S \leq k_S$.

The purpose of this paper is to prove a converse of the last example. However, before stating our result let us review briefly some of the uses of α–Lipschitz mappings in the literature. We do this in order to motivate an inequality which will be the main objective of our argument.

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The first result for α–Lipschitz mappings seems to be due to G. Darbo [2]. He proved the following result.

THEOREM D. *Let* $T : \Delta \to \Delta$ *be a continuous α–Lipschitz mapping defined on a closed bounded convex set in a Banach space* X. *If*

$$\alpha_T < 1,$$

then T *has a fixed point.*

In the same year M. A. Krasnosel'skii [4] published a proof of the following fixed point Theorem.

THEOREM K. *Let* R *be a compact continuous operator and let* S *be a Lipschitz continuous operator defined on a closed bounded convex set* Δ *in a Banach space* X. *If*

(i) $Rx + Sy \in \Delta$ *whenever* $x, y \in \Delta$, *and*

(ii) $k_S < 1$,

then the mapping $T = R + S$ *has a fixed point in* Δ.

Since, in Theorem K, one has $\alpha_T \leq k_S$, the following generalization of Theorem K is valid because it is a corollary of Theorem D.

THEOREM DK. *In Theorem* K *one can replace condition* (i) *with*

(i′) $Tx = Rx + Sx \in \Delta$ *whenever* $x \in \Delta$.

We will have more to say about these fixed point Theorems shortly, but let us now state the main result of this paper.

LOCAL DECOMPOSITION THEOREM. *Let* $T : U \to H$ *be a smooth α–Lipschitz mapping defined on an open set* U *in a Hilbert space* H. *Then for every* $x_0 \in U$ *and every* $\varepsilon > 0$ *there is a* $\delta > 0$, *a compact linear operator* R *and a Lipschitz continuous operator* S *such that*

$$T = R + S \tag{2}$$

in $B_\delta(x_0)$, *the* δ*–neighborhood of* x_0, *and moreover, the Lipschitz coefficient* k_S *for* S *satisfies*

$$\alpha_T \leq k_S \leq \alpha_T + 3\varepsilon. \tag{3}$$

We shall define the smoothness concept we require in the next section. Before doing that though, it should be emphasized that the important conclusion in this Theorem is Ineq. (3). This inequality implies, for example, that if $\alpha_T < 1$ then for smooth α–Lipschitz mappings one can choose S so that $k_S < 1$. In other words at least "locally" Theorem D and Theorem DK are the same. But, to re-emphasize, our result is only local. It is an interesting, but unsolved, problem to determine whether the Local Decomposition Theorem can be extended to an arbitrary closed bounded convex set Δ.

In Sections VI and VII we shall study the theory of a bounded linear mapping L on a Hilbert space. In these sections we shall present a formula for computing α_L in terms of the spectral properties of L. This formula not only plays a crucial role in our proof of the Local Decomposition Theorem, but seems to be interesting in itself.

II. Smooth Mappings

Let $T : U \to X$ be defined on an open set U in a Banach space X. We shall say that T is *smooth* if for every $x_0 \in U$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$Tx = Lx + Nx$$

in $B_\delta(x_0)$ where

(i) L is a bounded linear mapping on X,

(ii) $Tx_0 = Nx_0$, and

(iii) $\|Nx - Ny\| \leq \varepsilon \|x - y\|$ in $B_\delta(x_0)$.

The linear mapping L is, of course, the derivative of T at x_0. Also note that any smooth mapping on U has a continuous derivative on U, and that any C^2–function on U is smooth.

III. An Important Property of the α–Index

Some of the properties of the α–index have already been noted in the Introduction. We will also make use of the following property:

Let M be an infinite dimensional subspace of a Banach space X and let B_r denote a ball of radius r in M. Then $\alpha(B_r) = 2r$. Furthermore, if B is any bounded set in M with $B_r \subseteq B \subseteq B_s$, then

$$2r = \alpha(B_r) \leq \alpha(B) \leq \alpha(B_s) = 2s.$$

See [3] for a proof of these assertions.

IV. Reduction to a Linear Problem

It follows immediately from the definition that a smooth mapping is Lipschitz continuous. If $T = L + N$ in $B_\delta(x_0)$ where L is a bounded linear mapping, $Tx_0 = Nx_0$ and $\|Nx - Ny\| \leq \varepsilon \|x - y\|$, one then has

$$\alpha_L \leq \alpha_T + \varepsilon.$$

If one can show that the linear part L can be decomposed into $L = A + B$ where A is compact and where

$$k_B = \|B\| \leq \alpha_L + \varepsilon,$$

then by setting $R = A$ and $S = B + N$ we will have established Ineq. (3) and thus completed the proof of the Theorem. Indeed,

$$\alpha_T = \alpha_S \leq k_S \leq \|B\| + \varepsilon \leq \alpha_L + 2\varepsilon \leq \alpha_T + 3\varepsilon.$$

We shall now restrict our attention to the case where X is a Hilbert space H. The first step in our argument will involve an examination of the case where L is a positive self-adjoint operator, but before looking at this it will be convenient to review some of the spectral theory for self-adjoint operators.

V. A REVIEW OF SPECTRAL THEORY

Let $L : H \to H$ be a bounded positive self-adjoint operator on a Hilbert space H. That is, $0 \leq L$ or in terms of the inner product $0 \leq (Lx, x)$ for all $x \in H$. Then there exists a unique family of orthogonal projections E_λ and corresponding ranges M_λ, $-\infty < \lambda < \infty$, such that

(i) $\lambda \leq \mu \Rightarrow E_\lambda \leq E_\mu$,

(ii) $E_\lambda = 0$ for $\lambda < 0$,

(iii) $E_\lambda = I$ for $\lambda > \|L\|$,

(iv) E_λ is continuous from the right in the strong topology, and

(v) $L = \int_{-\infty}^{\infty} \lambda \, dE_\lambda = \int_{0^-}^{\|L\|} \lambda \, dE_\lambda$.

Let $\sigma(L)$ denote the spectrum of L. We recall that if λ is an isolated point of $\sigma(L)$, then λ is an eigenvalue. The next three lemmas are direct consequences of the Spectral Theorem, cf. [7] and we omit the proofs.

LEMMA 1. *Let τ be any nonnegative number with the property that $(\tau, +\infty) \cap \sigma(L)$ contains at most a finite number of points λ and that for each of these the null space $\mathfrak{n}(\lambda I - L)$ is finite dimensional. Let*

$$A = \int_{\tau^+}^{+\infty} \lambda \, dE_\lambda \quad \textit{and} \quad B = \int_{-\infty}^{\tau^+} \lambda \, dE_\lambda .$$

Then $L = A + B$ where A is compact and $\|B\| \leq \tau$. Furthermore if for any $\sigma < \tau$, the set $[\sigma, \tau] \cap \sigma(L)$ contains an infinite number of points, then there is an infinite dimensional subspace $M = M(\sigma)$ of H with the property that $\sigma I \leq L$ on M.

LEMMA 2. *Let τ satisfy the hypotheses of Lemma 1 and let A and B be determined accordingly. Then*

$$\alpha_L = \alpha_B \leq \|B\| \leq \tau .$$

LEMMA 3. *L is compact if and only if every nonzero $\lambda \in \sigma(L)$ is an isolated point of $\sigma(L)$ and $\mathfrak{n}(\lambda I - L)$ is finite dimensional.*

VI. Bounded Positive Self-Adjoint Operators

The primary purpose of this section is to compute α_L, in terms of the spectrum of L, when L is a bounded positive self-adjoint operator on a Hilbert space H. At the same time we shall present the decomposition of L prescribed in Section IV.

Let $L : H \to H$ be bounded positive self-adjoint operator. Define β_1 and β_2 as follows:

$$\beta_1 = \sup \{\lambda \in \sigma(L) : \lambda \text{ is an accumulation point of } \sigma(L)\}$$

$$\beta_2 = \max \{\lambda \in \sigma(L) : \lambda \text{ is isolated and } n(\lambda I - L) \text{ is infinite dimensional}\}$$

where we allow the values $-\infty$ if either of the above sets are empty. Let $\beta = \beta(L) = \max \{0, \beta_1, \beta_2\}$.

LEMMA 4. *Let* $L : H \to H$ *be a bounded positive self-adjoint operator on a Hilbert space* H. *Then* $\alpha_L = \beta(L)$. *Furthermore for every* $\varepsilon > 0$ *there is a decomposition* $L = A + B$ *where* A *is compact and* $\|B\| \leq \alpha_L + \varepsilon$.

Proof. We shall distinguish between four cases. In three of these cases we will see that B can be chosen independent of ε and that $\|B\| = \alpha_L$.

Case 1. $\beta = \beta_1 > 0$ and for every $\varepsilon > 0$ there are countably many eigenvalues in the interval $[\beta_1, \beta_1 + \varepsilon]$.

In this case we apply Lemma 1 with $\tau = \beta_1 + \varepsilon$ and $\sigma = \beta_1$. Since $\beta_2 < \beta_1$, it follows that the hypotheses of Lemma 1 are satisfied for this choice of τ and σ. Let A, B and M be determined by Lemma 1. Then A is compact and

$$\beta_1 \leq \sup \{\lambda \in \sigma(L) : \lambda \leq \beta_1 + \varepsilon\} = \|B\| \leq \beta_1 + \varepsilon . \tag{4}$$

Since $\beta_1 I \leq L$ on M it follows that $\beta_1 S \subseteq L(S)$, where S denotes the unit ball in M. Since M is infinite dimensional we have $2\beta_1 \leq \alpha(L(S))$, or $\beta_1 \leq \alpha_L$. By letting $\varepsilon \to 0$, it follows from Lemma 2 and Ineq. (4) that $\alpha_L = \beta_1$.

Case 2. $\beta = \beta_1 > 0$ and $(\beta_1, \infty) \cap \sigma(L)$ contains at most a finite number of points.

In this case we apply Lemma 1 with $\tau = \beta_1$ and σ being any real number with $0 < \sigma < \beta_1$. Since $\beta_2 < \beta_1$, it follows that the hypotheses of Lemma 1 are satisfied for this choice of τ and σ. Let A, B and M be determined accordingly. Then A is compact and

$$\|B\| = \sup \{\lambda \in \sigma(L) : \lambda \leq \beta_1\} = \beta_1 . \tag{5}$$

Since M is infinite dimensional and $\sigma I \leq L$ on M, it follows, as in Case 1, that $\sigma \leq \alpha_L$. By letting $\sigma \to \beta_1$ it then follows from Lemma 2 and Eqn. (5) that $\alpha_L = \beta_1$.

Case 3. $\beta = \beta_2 > 0$.

In this case we apply the first part of Lemma 1 with $\tau = \beta_2$. Since $\beta_1 < \beta_2$, the hypotheses of Lemma 1 are satisfied. With A and B given by Lemma 1 we see that A is compact and $\|B\| = \beta_2$. Furthermore, $L = \beta_2 I$ on the infinite dimensional space $M = n(\beta_2 I - L)$. It follows then, from the same argument used in Case 1, that $\beta_2 \leq \alpha_L$; and thus Lemma 2 implies that $\beta_2 = \alpha_L$.

Case 4. $\beta = 0$.

In this case it follows from Lemma 3 that L is compact. Hence $\alpha_L = 0 = \beta$. Now set $L = A$ and $B = 0$, to complete the proof.

VII. General Bounded Linear Operators

Now assume that L is an arbitrary bounded linear operator on the Hilbert space H. Then there is a polar decomposition $L = RU$, where R is a positive self-adjoint operator and U is a unitary operator, cf. [6] and [7]. Even though this decomposition may not be unique, the number $\beta(R)$ is well-defined and depends on L and not on the decomposition, since R is the positive square-root of LL^*.

LEMMA 5. *Let* $L: H \to H$ *be a bounded linear operator and let* $L = RU$ *be a polar decomposition of* L. *Then* $\alpha_L = \alpha_R = \beta(R)$. *Furthermore, for every* $\varepsilon > 0$ *there is a decomposition* $L = A + B$, *where* A *is compact and* $\|B\| \leq \alpha_L + \varepsilon$.

Proof: We apply Lemma 4 to R so that $R = P + Q$ where P is compact and $\|Q\| \leq \alpha_R + \varepsilon$. Since U is unitary it is clear that $\alpha_L = \alpha_R$. Furthermore it follows from Lemma 4 that $\alpha_R = \beta(R)$. Now set $A = PU$ and $B = QU$. Then $L = A + B$ where A is compact and

$$\|B\| = \|Q\| \leq \alpha_L + \varepsilon .$$

This completes the proof of Lemma 5 as well as the proof of the Local Decomposition Theorem.

VIII. Concluding Remarks

As we have seen, the problem of decomposing a given operator in order to establish Ineq. (3) reduces to the study of linear mappings when the given operator is smooth. Other Authors have studied linear α–Lipschitz mappings on a Banach space X.

A. Ambrosetti [1] has shown that if $L: X \to X$ is a linear α–Lipschitz mapping on a Banach space X, then for every $\varepsilon > 0$ one can write $L = A + B$, where A has finite dimensional range and r_B, the spectral radius of B, satisfies

$$r_B \leq \alpha_L + \varepsilon .$$

This result still leaves open the question of estimating the Lipschitz coefficient $\|B\|$, which still may be much longer than α_L.

Part of Lemmas 4 and 5 also follow from a Theorem of J. R. L. Webb [8], which gives a characterization of linear α–Lipschitz mappings with range in a Hilbert space. However, Webb's Theorem does not give a method for computing α_L in terms of spectral properties when L is linear.

Our theory leaves open a number of interesting unsolved problems for further research.

1) Can the main result be extended to Banach spaces or Frechét spaces?

2) The Local Decomposition Theorem depends on the base point x_0. Can one choose the decomposition so that it varies continuously with x_0, and perhaps more importantly, can one extend the local result to a global result?

References

[1] A. AMBROSETTI, *Proprietà spettrali di certi operatori lineari non compatti*, « Rend. Sem. Mat. Univ. Padova », *42*, 189–200 (1969).

[2] G. DARBO, *Punti uniti in trasformazioni a codominio non compatto*, « Rend. Sem. Mat. Univ. Padova », *24*, 84–92 (1955).

[3] M. FURI and A. VIGNOLI, *On a property of the unit sphere in a linear normed space*, « Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. », *18*, 333–334 (1970).

[4] M.A. KRASNOSEL'SKII, *Two remarks on the method of successive approximations*, « Usephi Mat. Nauk. », *10*, 123–127 (1955).

[5] K. KURATOWSKI, *Topologie*. Panstwowe Wydaw. Naukowe, Warsaw, 1958.

[6] A.W. NAYLOR and G.R. SELL, *Linear Operator Theory*. Holt, Rinehart and Winston, New York, 1971.

[7] F. RIESZ and B. SZ.–NAGY, *Functional Analysis*. Ungar, New York, 1955.

[8] J.R.L. WEBB, *On a characterization of k–set contractions*, « Atti Accad. Naz. Lincei, Rend. Cl. Sci. fis., mat. nat. », *50*, 686–689 (1971).