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On Nörlund summability of Jacobi Series

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Analisi matematica. — *On Nörlund summability of Jacobi Series.* Nota di R. S. Choudhary, presentata (*) dal Socio G. Sansone.

Riassunto. — L'Autore prova un Teorema sulla sommabilità secondo Nörlund delle serie di Fourier–Jacobi corrispondente ad un Teorema di T. Singh per le serie trigonometriche di Fourier.

1. Let Σa_n be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of Constants, real or complex and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n .$$

The sequence to sequence transformation; Viz.

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{\nu=n} p_{n-\nu} S_\nu = \frac{1}{P_n} \sum_{\nu=0}^{\nu=n} p_\nu S_{n-\nu} , \quad P_n \neq 0 , \tag{1.1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$, generated by the sequence of constants $\{p_n\}$. The series Σa_n is said to be summable (N , p_n) to the sum S if $\lim t_n$ exists and equals S.

The conditions of regularity of the method of summability (N , p_n) defined by (1.1) are

$$\lim_{n \to \infty} \frac{p_n}{P_n} = 0 \tag{1.2}$$

$$\lim_{n \to \infty} \sum_{k=0}^{k=n} |p_k| = O(P_n) . \tag{1.3}$$

If $\{p_n\}$ is a real non negative sequence then the condition (1.3) is automatically satisfied and in that case (1.2) is the necessary and sufficient condition for the regularity of the method of summation.

Two important particular cases of (N , p_n) summability are (*i*) Harmonic summability when $p_n = \frac{1}{n+1}$ and (*ii*) Cesàro summability when

$$p_n = \binom{n+\delta-1}{\delta-1} , \ \delta > 0 .$$

2. Let $f(x)$ be a function defined in the closed interval $[-1 , +1]$ such that the function

$$(1-x)^\alpha (1+x)^\beta f(x) \in L [-1 , +1]$$

(*) Nella seduta dell'11 marzo 1972.

and $\alpha > -1$, $\beta > -1$. The Fourier Jacobi Series expansion corresponding to $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x), \tag{2.1}$$

where

$$a_n = \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\,2^{\alpha+\beta+1}} \tag{2.2}$$

$$\cdot \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\, P_n^{(\alpha,\beta)}(x)\, dx$$

and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials.

Dealing with (N, p_n) summability of Jacobi Series Gupta (2) proved the following Theorem.

Write

$$F(\varphi) = [f(\cos\varphi) - A]\left(\sin\frac{\varphi}{2}\right)^{2\alpha+1}\left(\cos\frac{\varphi}{2}\right)^{2\beta+1}.$$

THEOREM. *Let $\{p_n\}$ be a non negative non increasing sequence such that*

$$\sum_{a}^{n} \frac{P_k}{k^{\alpha+3/2}\log k} = O\left(\frac{P_n}{n^{\alpha+1/2}}\right), \tag{2.3}$$

a being fixed positive Integer and

$$\sum_{n} \frac{n^{\alpha+1/2}}{P_n} < \infty. \tag{2.4}$$

If

$$\psi(t) \equiv \int_0^t |F(\varphi)|\, d\varphi = o\left(\frac{t^{2\alpha+2}}{\log 1/t}\right) \tag{2.5}$$

then the series is summable (N, p_n) at the point $x = +1$ to the sum A provided $-1/2 \le \alpha < 1/2$, $\beta > -1/2$ and the antipole condition

$$\int_{-1}^{b} (1+x)^{\beta/2-3/4}\, |f(x)|\, dx < \infty, \tag{2.6}$$

b fixed, is satisfied.

The object of this paper is to remove restriction (2.3) and replace the condition (2.5) by a more general condition.

3. We prove the following Theorem:

THEOREM. *Let* (N, p_n) *be a regular Nörlund method defined by a real non negative monotonic non increasing sequence of coefficients* $\{p_n\}$ *such that* $P_n \to \infty$ *as* $n \to \infty$ *then if*

$$\psi(t) = o\left[\frac{p(1/t)}{P(1/t)} t^{2\alpha+1}\right], \quad \text{as } t \to 0 \tag{3.1}$$

and

$$\sum_n \frac{n^{\alpha+1/2}}{P_n} < \infty, \tag{3.2}$$

then the series (2.1) *is summable* (N, p_n) *at the point* $x = 1$ *to the sum* A *provided* $-1/2 \leq \alpha < 1/2$, $\beta > -1/2$ *and the antipole condition*

$$\int_{-1}^{b} (1 + x)^{\beta/2-3/4} |f(x)| \, dx < \infty, \tag{3.3}$$

b *fixed, is satisfied.*

4. The following Lemmas are pertinent to the Proof of our Theorem.

LEMMA 1 (Szegö [6], p. 167) *for* α, β *arbitrary and real and* C *a fixed positive constant*

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2} O(n^{-1/2}), & \text{for } c/n \leq \theta \leq \pi/2, \\ O(n^\alpha), & \text{for } 0 \leq \theta \leq c/n. \end{cases} \tag{4.1}$$

LEMMA 2 (Szegö [6], p. 196). *If* $\alpha > -1$, $\beta > -1$, $c/n \leq \theta \leq \pi - c/n$

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2} u(\theta) \left[\cos(N\theta + \gamma) + \frac{O(1)}{n \sin\theta}\right] \tag{4.2}$$

where

$$u(\theta) = \frac{1}{\sqrt{\pi}} \left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2} \left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}, \quad N = n + \frac{\alpha+\beta+1}{2}, \quad \gamma = -\left(\alpha + \frac{1}{2}\right)\frac{\pi}{2}.$$

LEMMA 3 (McFadden [5]). *If* $\{p_n\}$ *is a non negative and non increasing sequence then*

$$n p_n \leq P_n.$$

LEMMA 4 (Gupta [2]). *Let*

$$N_n(\varphi) = \frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{k=n} p_k \lambda_{n-k} P_{n-k}^{(\alpha+1,\beta)}(\cos\varphi)$$

where

$$\lambda_n = \frac{2^{-\alpha-\beta-1}\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \cong \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)} n^{\alpha+1}.$$

(4.3) *For* $0 \leq \varphi \leq 1/n$; $|N_n(\varphi)| = O(n^{2\alpha+2})$.

LEMMA 5 (Gupta [2]). *For* $1/n \leq \varphi \leq \pi - 1/n$

$$(4.4) \qquad |N_n(\varphi)| = \frac{1}{P_n} O\left[n^{\alpha+1/2} P(1/\varphi) \Big/ \left(\sin\frac{\varphi}{2}\right)^{\alpha+3/2} \left(\cos\frac{\varphi}{2}\right)^{\beta+1/2}\right] + O\left[n^{\alpha-1/2} \Big/ \left(\sin\frac{\varphi}{2}\right)^{\alpha+5/2} \left(\cos\frac{\varphi}{2}\right)^{\beta+3/2}\right].$$

LEMMA 6 (Gupta [2]). *The antipole condition Viz. the condition*

$$(4.5) \qquad \int_{-1}^{b} (1+x)^{\beta/2-3/4} |f(x)|\, dx < \infty,$$

implies that

$$(4.6) \qquad \int_{a}^{\pi} \left(\cos\frac{\varphi}{2}\right)^{\beta-1/2} |f(\cos\theta) - A|\, d\theta < \infty,$$

which further implies that

$$(4.7) \qquad \int_0^{1/n} t^{\beta-1/2} |f(-\cos t) - A|\, dt = o(1).$$

5. The n-th partial sum of the series (2.1) at the end point $x = 1$ is given by (Obrechkoff, [1], p. 99)

$$(5.1) \qquad S_n(1) = 2^{\alpha+\beta} \int_0^{\pi} \left(\sin\frac{\varphi}{2}\right)^{2\alpha} \left(\cos\frac{\varphi}{2}\right)^{2\beta} f(\cos\varphi)\, S_n(1, \cos\varphi) \sin\varphi\, d\varphi$$

where $S_n(1, \cos\varphi)$ denote the n-th partial sum of the series

$$(5.2) \qquad \sum_u \frac{P_u^{(\alpha,\beta)}(1)\, P_u^{(\alpha,\beta)}(\cos\varphi)}{g_u}$$

and

$$(5.3) \qquad g_u = \frac{2^{\alpha+\beta+1}\,\Gamma(u+\alpha+1)\,\Gamma(u+\beta+1)}{(2u+\alpha+\beta+1)\,\Gamma(u+1)\,\Gamma(u+\alpha+\beta+1)}.$$

As shown by (Rau [4])

$$S_n(1, \cos\varphi) = \lambda_n P_n^{(\alpha+1,\beta)}(\cos\varphi),$$

where λ_n is as defined in Lemma 4.

Consequently

$$(5.4) \qquad S_n(1) - A = 2^{\alpha+\beta+1} \lambda_n \int_0^{\pi} \left(\sin\frac{\varphi}{2}\right)^{2\alpha+1} \left(\cos\frac{\varphi}{2}\right)^{2\beta+1} \times [f(\cos\varphi) - A]\, P_n^{(\alpha+1,\beta)}(\cos\varphi)\, d\varphi$$

$$= 2^{\alpha+\beta+1} \lambda_n \int_0^{\pi} F(\varphi)\, P_n^{(\alpha+1,\beta)}(\cos\varphi)\, d\varphi.$$

The Nörlund mean (N, p_n) of the series (2.1) at the point $x = 1$ is given by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{k=n} p_k \, S_{n-k}(1) \tag{5.5}$$

or

$$t_{n-A} = \frac{1}{P_n} \sum_{k=0}^{k=n} p_k \, [S_{n-k}(1) - A]$$

$$= \frac{1}{P_n} \sum_{k=0}^{k=n} p_k \, 2^{\alpha+\beta+1} \lambda_{n-k} \int_0^\pi F(\varphi) \, P_{n-k}^{(\alpha+1,\beta)}(\cos\varphi)$$

$$= \int_0^\pi F(\varphi) \, N_n(\varphi) \, d\varphi \, ,$$

where

$$N_n(\varphi) = \frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{k=n} p_k \, \lambda_{n-k} \, P_{n-k}^{(\alpha+1,\beta)}(\cos\varphi) \, .$$

In order to prove the Theorem, therefore we have to show that

$$I(\varphi) = \int_0^\pi F(\varphi) \, N_n(\varphi) \, d\varphi = o(1), \quad \text{as} \quad n \to \infty \, .$$

Write

$$I(\varphi) = \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-1/n} + \int_{\pi-1/n}^{\pi} \tag{5.6}$$

δ being a suitable constant.

$$I_1 = \int_0^{1/n} F(\varphi) \, N_n(\varphi) \, d\varphi$$

$$| I_1 | = O\left[\int_0^{1/n} (n^{2\alpha+2}) \, | F(\varphi) | \, d\varphi \right]$$

$$= O(n^{2\alpha+2}) \times o\left(\frac{p_n \dfrac{1}{n^{2\alpha+1}}}{P_n} \right) ,$$

Using Lemma 4 and hypothesis

$$= o(n^{2\alpha+1}) \cdot \left(\frac{n p_n}{P_n} \right) \cdot \frac{1}{n^{2\alpha+1}} \tag{5.7}$$

$$= o(1) \, , \qquad \text{using Lemma 3.}$$

Making use of the result of Lemma 5

$$|I_2| = O\left[\int_{1/n}^{\delta} |F(\varphi)| \frac{n^{\alpha+1/2}}{P_n} P(1/\varphi)\left(\sin\frac{\varphi}{2}\right)^{-\alpha-3/2} d\varphi\right]$$

$$+ O\left[\int_{1/n}^{\delta} |F(\varphi)|\, n^{\alpha-1/2}\left(\sin\frac{\varphi}{2}\right)^{-\alpha-5/2} d\varphi\right]$$

$$= J_1 + J_2, \text{ say.}$$

$$J_1 = O\left(\frac{n^{\alpha+1/2}}{P_n}\right)\int_{1/n}^{\delta} \frac{|F(\varphi)|\, P(1/\varphi)}{\varphi^{\alpha+3/2}} d\varphi$$

$$= O\left(\frac{n^{\alpha+1/2}}{P_n}\right)\left[\left\{\psi(\varphi)\frac{P(1/\varphi)}{\varphi^{\alpha+3/2}}\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta} \psi(\varphi)\frac{d}{d\varphi}\left\{\frac{P(1/\varphi)}{\varphi^{\alpha+3/2}}\right\} d\varphi\right]$$

$$= O\left(\frac{n^{\alpha+1/2}}{P_n}\right)\left[o\left(\frac{p(1/\varphi)}{P(1/\varphi)}\varphi^{2\alpha+1}\right)\frac{P(1/\varphi)}{\varphi^{\alpha+3/2}}\right]_{1/n}^{\delta}$$

$$+ O\left(\frac{n^{\alpha+1/2}}{P_n}\right)\left[\int_{1/n}^{\delta} o\left(\frac{p(1/\varphi)}{P(1/\varphi)}\varphi^{2\alpha+1}\right)\frac{d}{d\varphi}\left\{\frac{P(1/\varphi)}{\varphi^{\alpha+3/2}}\right\} d\varphi\right]$$

$$= J_{1\cdot 1} + J_{1\cdot 2}, \text{ say.}$$

Now

$$J_{1\cdot 1} = O\left(\frac{n^{\alpha+1/2}}{P_n}\right)\left[o\, p(1/\varphi)\, \varphi^{\alpha-1/2}\right]_{1/n}^{\delta}$$

$$= o\left(\frac{n^{\alpha+1/2}}{P_n}\right) + o\left(\frac{n^{\alpha+1/2}}{P_n}\right) p_n\, n^{-\alpha+1/2}$$

$$= o\left(\frac{n^{\alpha+1/2}}{P_n}\right) + o\left(\frac{n p_n}{P_n}\right)$$

$$= o(1), \qquad \text{by (3.2) and Lemma 3.}$$

Consider the integral

$$\int_{1/n}^{\delta} \frac{p(1/\varphi)}{P(1/\varphi)}\varphi^{2\alpha+1}\left|\frac{d}{d\varphi}\left\{\frac{P(1/\varphi)}{\varphi^{\alpha+3/2}}\right\}\right| d\varphi$$

$$= \int_{1/\delta}^{n} \frac{p(x)}{P(x)}\frac{1}{x^{2\alpha+1}}\left|\frac{d}{dx}\left\{P(x)\, x^{\alpha+3/2}\right\}\right| dx$$

$$= O(1)\left[\int_{1/\delta}^{a} \frac{p(x)}{P(x)}\frac{1}{x^{2\alpha+1}}\left|\frac{d}{dx}\left\{P(x)\, x^{\alpha+3/2}\right\}\right|\right] dx +$$

$$+ O(1)\left[\int_{a}^{n} \frac{p(x)}{P(x)}\frac{1}{x^{2\alpha+1}}\left|\frac{d}{dx}\left\{P(x)\, x^{\alpha+3/2}\right\}\right|\right] dx$$

where $a = [1/\delta] + 1$,

$$= O(1) + O\left[\sum_a^n \frac{p(k)}{P(k)} \frac{1}{k^{2\alpha+1}} \left|\Delta\{k^{\alpha+3/2} P(k)\}\right|\right]$$

$$= O(1) + O\left[\sum_a^n \frac{p(k)}{P(k)} \frac{1}{k^{2\alpha+1}} \{k^{\alpha+3/2} P(k) - (k+1)^{\alpha+3/2} P(k+1)\}\right]$$

$$= O(1) + O\left[\sum_a^n \frac{p(k)}{P(k)} \frac{1}{k^{\alpha+1/2}} P(k) + \sum_a^n \frac{p(k)}{P(k)} \frac{1}{k^{\alpha+1/2}} \{(k+1)\, p(k+1)\}\right]$$

$$= O(1) + O\left[\sum_a^n \frac{P(k)}{k^{\alpha+3/2}}\right] + O\left[\sum_a^n \frac{P(k)}{k^{\alpha+3/2}}\right]$$

$$= O\left[\frac{P(n)}{n^{\alpha+1/2}}\right].$$

And

$$\int_{1/n}^{\delta} |F(\varphi)|\, \varphi^{-\alpha-5/2}\, d\varphi$$

$$= \left[\varphi^{-\alpha-5/2}\, o\left(\frac{p(1/\varphi)}{P(1/\varphi)}\, \varphi^{2\alpha+1}\right)\right]_{1/n}^{\delta} + o\left[\int_{1/n}^{\delta} \varphi^{-\alpha-7/2} \left(\frac{p(1/\varphi)}{P(1/\varphi)}\, \varphi^{2\alpha+1}\right) d\varphi\right]$$

$$= o(1) + o\left[\frac{1}{n^{\alpha-1/2}} \frac{np_n}{P_n}\right] + o\left[\int_{1/n}^{\delta} \varphi^{\alpha-5/2} \frac{p(1/\varphi)}{P(1/\varphi)}\, d\varphi\right].$$

Now

$$\int_{1/n}^{\delta} \varphi^{\alpha-5/2} \frac{p(1/\varphi)}{P(1/\varphi)}\, d\varphi$$

$$= \int_{1/\delta}^{n} x^{1/2-\alpha} \frac{p(x)}{P(x)}\, dx$$

$$= O(1)\left[\int_{1/\delta}^{n} x^{-\alpha-1/2}\, dx\right]$$

$$= O(n^{1/2-\alpha}).$$

Therefore

$$\int_{1/n}^{\delta} |F(\varphi)|\, \varphi^{-\alpha-5/2}\, d\varphi$$

$$= o(1) + o\left[\frac{1}{n^{\alpha-1/2}} \frac{np_n}{P_n}\right] + o(n^{1/2-\alpha}).$$

Thus we have

(5.8) $$|J_1| = o(1)$$

and

(5.9) $$|J_2| = o(1).$$

Coming to I_3 we have

$$|I_3| = O\int_{\delta}^{\pi-1/n} |F(\varphi)| \left(\frac{n^{\alpha+1/2}}{P_n}\right) P(1/\varphi) \frac{d\varphi}{\left(\sin\frac{\varphi}{2}\right)^{\alpha+3/2}\left(\cos\frac{\varphi}{2}\right)^{\beta+1/2}}$$

$$+ O(n^{\alpha-1/2})\int_{\delta}^{\pi-1/n} |F(\varphi)| \frac{d\varphi}{\left(\sin\frac{\varphi}{2}\right)^{\alpha+5/2}\left(\cos\frac{\varphi}{2}\right)^{\beta+3/2}}$$

$$= O\left(\frac{n^{\alpha+1/2}}{P_n}\right)\int_{\delta}^{\pi-1/n} |f(\cos\varphi) - A| \left(\cos\frac{\varphi}{2}\right)^{\beta-1/2} \cos\frac{\varphi}{2}\, d\varphi +$$

$$+ O(n^{\alpha-1/2})\int_{\delta}^{\pi-1/n} |f(\cos\varphi) - A| \left(\cos\frac{\varphi}{2}\right)^{\beta-1/2} d\varphi$$

$$= O\left(\frac{n^{\alpha+1/2}}{P_n}\right) + O(n^{\alpha-1/2}), \quad \text{by (4.6)}$$

(5.10) $= o(1)$ by hypothesis and from the condition $\alpha < 1/2$.

We finally come to I_4

$$I_4 = \int_{\pi-1/n}^{\pi} F(\varphi) \frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{k=n} p_k \lambda_{n-k} P_{n-k}^{(\alpha+1,\beta)}(\cos\varphi)\, d\varphi$$

substituting $\varphi = \pi - \theta$ we obtain

$$I_4 \cong \sum_{k=0}^{k=n} \frac{2^{\alpha+\beta+1}}{P_n} p_k \lambda_{n-k} \times \int_0^{1/n} F(\pi-\theta) P_{n-k}^{(\beta,\alpha+1)}(\cos\theta)\, d\theta$$

$$= O\left(\frac{1}{P^n}\right)\sum_{k=0}^{k=n} p_k (n-k)^{\alpha+1}\int_0^{1/n} |F(\pi-\theta)| \cdot O(n-k)^{\beta}\, d\theta$$

$$= O(n^{\alpha+\beta+1})\int_0^{1/n} |f(-\cos\theta) - A| (\sin\theta/2)^{2\beta+1}(\cos\theta/2)^{2\alpha+1}\, d\theta$$

$$= O(n^{\alpha+\beta+1})\int_0^{1/n} |f(-\cos\theta) - A|\, \theta^{2\beta+1}\, d\theta.$$

Writing

$$u(\varphi) = \int_0^{\varphi} t^{\beta-1/2} \left| f(-\cos t) - A \right| dt$$

$$\left| I_4 \right| = O(n^{\alpha+\beta+1}) \int_0^{1/n} \left| f(-\cos\theta) - A \right| \theta^{\beta-1/2}\, \theta^{\beta+3/2}\, d\theta$$

$$= O(n^{\alpha+\beta+1}) \left[t^{\beta+3/2}\, u(t) \right]_0^{1/n} + O(n^{\alpha+\beta+1}) \cdot \int_0^{1/n} t^{\beta+1/2}\, u(t)\, dt$$

$$= o(n^{\alpha-1/2}) + o(1) + O(n^{\alpha+\beta+1}) \cdot o(n^{-\beta-3/2})$$

$$= o(n^{\alpha-1/2}) + o(1), \quad \text{by relation (4.7)}$$

$$= o(1). \tag{5.11}$$

Combining (5.6), (5.7), (5.8), (5.9), (5.10), (5.11) we have $I(\varphi) = o(1)$.

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