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Banach algebras of the type $l_1(S, A)$

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Matematica. — *Banach algebras of the type* l_1 (S, A). Nota di OLUSOLA AKINYELE, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Sia G un gruppo Abeliano localmente compatto e A un'algebra di Banach commutativa. Hausner [3] e Johnson [6] hanno discusso gli omomorfismi complessi dell'algebra di Banach $B'(G, A)$, costituita da tutte le funzioni di Bochner integrabili definite in un gruppo Abeliano localmente compatto avente valori in A. È noto che lo spazio di tutti gli ideali regolari massimi di $B'(G, A)$ è omeomorfo nella topologia del prodotto al prodotto Cartesiano di $\mathfrak{M}(A)$ e $\hat{G}$, dove $\mathfrak{M}(A)$ indica lo spazio di tutti gli ideali regolari massimi di A e $\hat{G}$ è il gruppo di caratteri di G. Johnson ha dimostrato che nel caso in cui A stesso è l'algebra di gruppo $L_1(H)$ di un gruppo localmente compatto H, $B^1(G, L_1(H))$ è isometrico e isomorfo all'algebra di gruppo $L_1(G \times H)$. Lo scopo del presente lavoro è di generalizzare questi risultati all'algebra di Banach $l_1(S, A)$ discussa in [1], dove S è un semigruppo commutativo discreto.

§ 1. INTRODUCTION

Let G be a locally compact Abelian group and A a commutative Banach algebra. Hausner [3] and Johnson [6] have discussed the complex homomorphisms of the Banach algebra $B^1(G, A)$, consisting of all of Bochner integrable functions defined on a locally compact Abelian group with values in A. It is known [3,6] that the space of all regular maximal ideals of $B^1(G, A)$ is homeomorphic with the Cartesian product of $\mathfrak{M}(A)$ and $\hat{G}$ in the product topology, where $\mathfrak{M}(A)$ denotes the space of all regular maximal ideals of A and $\hat{G}$ is the group of characters of G. Johnson [6] has shown that in the case where A itself is the group algebra $L_1(H)$ of a locally compact group H, $B^1(G, L_1(H))$ is isometric and isomorphic to the group algebra $L_1(G \times H)$.

The aim of this paper is to generalise the above results to the case where G is a semigroup. Throughout this paper S will be a discrete commutative semigroup and A a commutative Banach algebra. In [1], the Author has studied the convolution algebra $l_1(S, A)$, which specializes to $l_1(S)$ of Hewitt and Zuckerman [4], if A is taken to be the complex numbers. In that paper, Theorem 3.3 and Corollary 3.4 show that the maximal regular ideal space of $l_1(S, A)$ can be identified with an ordered pair (M, χ), where M is a maximal regular ideal of the Banach algebra A and χ is a semicharacter of S, just as in the case where S is a group, that is, $B^1(G, A)$ [3, 6]. This identification is given by the formula:

$$J_{(M,\chi)}(f) = \sum_{x \in S} \varphi_M(f(x))\,\chi(x), \qquad f \in l_1(S, A),$$

(*) Nella seduta del 13 maggio 1972.

where φ_M is a complex-valued homomorphism of A. In section 2, we show that the identification obtained in [1] is in fact a homeomorphism. In section 3, we show that in the case where A is an l_1–algebra, that is, if $A = l_1(G)$, then $l_1(S, l_1(G))$ is isometric and isomorphic with the l_1–algebra $l_1(S \times G)$. As a corollary we show that $\widehat{S \times G}$ is homeomorphic with $\hat{S} \times \hat{G}$.

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§ 2. The topology of $\mathfrak{M}(l_1(S, A))$

Our main interest in this section is to obtain a homeomorphism between the space $\mathfrak{M}(l_1(S, A))$ of maximal regular ideals of $l_1(S, A)$ and the space $\mathfrak{M}(A) \times \hat{S}$ with the product topology.

We begin with a description of the topologies of $\hat{S}$ and $\mathfrak{M}(A)$.

2.1. According to the well-known Gelfand theory for Commutative Banach algebras [cfr., 7, Chapters IV and V], the space of regular maximal ideals of $l_1(S)$ may be identified with the space of all homomorphisms of $l_1(S)$ onto the complex numbers [7, Theorem 23 A]. By [4, Theorem 2.7] we can identify the space of regular maximal ideals of $l_1(S)$ with the set of all semicharacters of S. We associate with every $f \in l_1(S)$ its Fourier transform $\hat{f}$, which is the complex function of $\hat{S}$, whose value at $\chi \in \hat{S}$ is given by $\hat{f}(\chi) = \sum_{x \in S} f(x)\chi(x)$. We now define a topology for $\hat{S}$ in such a way that every function $\hat{f}$ is continuous. The most natural way of doing this is to introduce the weakest topology generated by all functions $\hat{f}$. This is the weakest topology (i.e. the topology with the smallest family of open sets) on $\hat{S}$ relative to which every function $\hat{f}$ is continuous. This topology is called the Gelfand topology of $\hat{S}$. It is well-known [7, Theorem 19B] that $\hat{S}$ is a locally compact Hausdorff space and that all functions $\hat{f}$ vanish at infinity in $\hat{S}$. $\hat{S}$ is therefore a locally compact Hausdorff space under the Gelfand topology.

2.2. Suppose a is a fixed element in A then $\varphi_M(a)$ defines a complex-valued function $\hat{a}$ on the set $\mathfrak{M}(A)$ of all regular maximal ideals of A, by $\hat{a}(M) = \varphi_M(a)$. Suppose $\mathfrak{M}(A)$ is given the weak topology which makes the functions $\hat{a}$ all continuous. By Theorem 19B of [7], $\mathfrak{M}(A)$ is locally compact and the functions $\hat{a}$ all vanish at infinity in $\mathfrak{M}(A)$. The product topology of $\mathfrak{M}(A)$ and $\hat{S}$ is locally compact [5, Theorem 2–52]. We are now in a position to prove the following.

2.3. Theorem. *The space $\mathfrak{M}(l_1(S, A))$ of all regular maximal ideals of $l_1(S, A)$ with the weak topology is homeomorphic with the space $\mathfrak{M}(A) \times \hat{S}$ with the product topology.*

Proof. By Corollary 3.4 of [1], $\exists a$ 1 — 1 correspondence between $\mathfrak{M}(l_1(S, A))$ and $\mathfrak{M}(A) \times \hat{S}$. For $f \in l_1(S, A)$ define $\hat{f}$ on $\mathfrak{M}(A) \times \hat{S}$

by setting $\hat{f}(M, \chi) = J_{(M,\chi)}(f)$. The weak topology of $\mathfrak{M}(l_1(S, A))$ is the topology induced on $\mathfrak{M}(A) \times \hat{S}$ by the family of functions $\mathfrak{F} = \{\hat{f} \mid f \in l_1(S, A)\}$ which are defined on $\mathfrak{M}(A) \times \hat{S}$. For each positive integer n, $\{f_i\}_{i=1}^{n} \subset l_1(S)$ and $\{a_i\}_{i=1}^{n} \subset A$, there exists a function $\hat{h}$ defined on $\mathfrak{M}(A) \times \hat{S}$ by $\hat{h}(M, \chi) = \widehat{\sum_{i=1}^{n} a_i f_i}(M, \chi)$. Let $\mathfrak{J}$ be the family of functions so defined, then $\mathfrak{J} \subset \mathfrak{F}$. It will be shown that $\mathfrak{J}$ is dense in $\mathfrak{F}$ in the uniform-norm topology which will in turn imply that the supnorm topologies induced on $\mathfrak{M}(A) \times \hat{S}$ by $\mathfrak{J}$ and $\mathfrak{F}$ are identical. In fact, suppose $f \in l_1(S, A)$, then $\exists f_1, f_2, \cdots, f_n \in l_1(S)$ and $a_1, \cdots, a_n \in A$ such that $\left\| f - \sum_{i=1}^{n} a_i f_i \right\| < \epsilon$, for a given $\varepsilon > 0$. Then clearly,

$$\left| \hat{f}(M, \chi) - \widehat{\sum_{i=1}^{n} a_i f_i}(M, \chi) \right| \leq \left\| f - \sum_{i=1}^{n} a_i f_i \right\| < \varepsilon .$$

Hence

$$\sup_{(M,\chi) \in \mathfrak{M}(A) \times \hat{S}} \left| \hat{f}(M, \chi) - \widehat{\sum_{i=1}^{n} a_i f_i}(M, \chi) \right| \leq \varepsilon .$$

The family of functions $\mathfrak{J}$ satisfy the following properties:

(I) The functions in $\mathfrak{J}$ separate points of $\mathfrak{M}(A) \times \hat{S}$ [cfr. 1, Cor. 3.4].

(II) Functions in $\mathfrak{J}$ are continuous over $\mathfrak{M}(A) \times \hat{S}$ in the product topology.

In fact if $f \in l_1(S)$, $a \in A$ and (M_0, χ_0) is a fixed point of $(A) \times \hat{S}$ and $\varepsilon > 0$, then

$$| \hat{af}(M, \chi) - \hat{af}(M_0, \chi_0) | \leq \| a \| \, | \hat{f}(\chi) - \hat{f}(\chi_0) | + | \hat{a}(M) - \hat{a}(M_0) | \, \| f \|_{l_1(S)} .$$

If $(M, \chi) \in \cup M_0 \times \cup (\chi_0)$, where $\cup (M_0)$ and $\cup (\chi_0)$ are neighbourhoods of M_0 and χ_0 in $\mathfrak{M}(A)$ and $\hat{S}$ respectively $\ni | \hat{a}(M) - \hat{a}(M_0) | < \frac{\varepsilon}{2 \| f \|_{l_1(s)}}$ and $| \hat{f}(\chi) - \hat{f}(\chi_0) | < \frac{\varepsilon}{2 \| a \|}$, then $| \hat{af}(M, \chi) - \hat{af}(M_0, \chi_0) | < \varepsilon$. Hence $\hat{af}$ is continuous at (M_0, χ_0) and by Remark 1 of [1] the functions in $\mathfrak{J}$ are continuous on $\mathfrak{M}(A) \times \hat{S}$.

(III) Not all functions in $\mathfrak{J}$ vanish at a fixed point of $\mathfrak{M}(A) \times \hat{S}$.

(IV) Each function in $\mathfrak{J}$ vanishes at infinity in $\mathfrak{M}(A) \times \hat{S}$.

In fact, suppose $a_1, a_2, \cdots a_n \in A$ and $f_1, f_2, \cdots, f_n \in l_1(s)$:

Let o_i, $i = 1, 2, \cdots n$ be the compact sets outside which $\hat{a}_i(M) = \varphi_M(a_i)$ are small and Q_i, $i = 1, 2, \cdots, n$ the compact sets outside which $\hat{f}_i(\chi)$ are small. Then $\exists \delta > 0$ such that $| \hat{a}_i(M) | = | \varphi_M(a_i) | < \delta$ if $M \notin \bigcup_{i=1}^{n} o_i$ and $| \hat{f}_i(\chi) | < \delta$ if $\chi \notin \bigcup_{i=1}^{n} Q_i \cdot \bigcup_{i=1}^{n} o_i \times \bigcup_{i=1}^{n} Q_i$ is a compact subset of $\mathfrak{M}(A) \times \hat{S}$ by Tyconoff's Theorem [5, Theorem 1–28].

Hence if $(M, \chi) \notin \bigcup_{i=1}^{n} o_i \times \bigcup_{i=1}^{n} Q_i$, then $M \notin \bigcup_{i=1}^{n} o_i$ or $\chi \notin \bigcup_{i=1}^{n} Q_i$.

$$\left|\widehat{\sum_{i=1}^{n} a_i f_i}(M, \chi)\right| = \left|\sum_{i=1}^{n} \hat{f}_i(\chi)\, \hat{a}_i(M)\right| \leq n \sup_{1\leq i\leq n} \sup_{\chi} |\hat{f}_i(\chi)| \sup_{1\leq i\leq n} |\hat{a}_i(M)| .$$

Let $k_1 = \sup_{1\leq i\leq n} \sup_{n} |\hat{f}_i(\chi)|$ and $k_2 = \sup_{1\leq i\leq n} |\hat{a}_i(M)|$. Then $\left|\widehat{\sum_{i=1}^{n} a_i f_i}(M, \chi)\right| \leq$ $\leq n k_1 k_2$. Given $\varepsilon > 0$, then for $M \notin \bigcup_{i=1}^{n} o_i$, $\left|\widehat{\sum_{i=1}^{n} a_i f_i}(M, \chi)\right| \leq \varepsilon$ provided $\delta \leq \frac{\varepsilon}{n k_1}$. On the other hand if $\chi \notin \bigcup_{i=1}^{n} Q_i$, then $\left|\widehat{\sum_{i=1}^{n} a_i f_i}(M, \chi)\right| < \varepsilon$ provided that $\delta \leq \frac{\varepsilon}{n k_2}$. Hence if we choose $\delta \ni \delta \leq \min\left(\frac{\varepsilon}{n k_1}, \frac{\varepsilon}{n k_2}\right)$, $\left|\widehat{\sum_{i=1}^{n} a_i f_i}(M, \chi)\right| \leq \varepsilon$ for $(M, \chi) \notin \bigcup_{i=1}^{n} o_i \times \bigcup_{i=1}^{n} Q_i$. An appeal to a theorem in general point set topology [7, pp. 12] completes the proof.

§ 3. The case where A is an l_1–algebra

In this section, we shall study the Banach algebra $l_1(S, A)$ in the special case where A is an l_1–algebra. Let $A = l_1(G)$, where G is a discrete commutative semigroup. A function in $l_1(S, l_1(G))$ takes at each $x \in S$ a value which is an element of $l_1(G)$. In the case where S and G are locally compact Abelian groups, Johnson [6] has shown that $B^1(S, L_1(G))$ is isometric and isomorphic with the group algebra $L_1(S \times G)$. We wish to show in this section that the result of Johnson can be generalized to the algebra $l_1(S, l_1(G))$.

3.1. We begin by defining a dense subset of $l_1(S, l_1(G))$. Consider the subset $\mathfrak{D}$ of $l_1(S, l_1(G))$ consisting of all finite linear combinations of elements of the form hk, where $hk(x) = h(x)\cdot k$ for $h \in l_1(S)$ and $k \in l_1(G)$, that is, if $f \in \mathfrak{D}$, $f = \sum_{i=1}^{n} h_i k_i$. The set $\mathfrak{D}$ contains the simple functions and hence it is dense in $l_1(S, l_1(G))$.

3.2. Let S and G be commutative semigroups. Denote by $S \times G$ the set $\{(x, y) : x \in S, y \in G\}$. Define multiplication in $S \times G$ as follows:

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2, y_1 y_2) \qquad (x_1, x_2 \in S ; y_1, y_2 \in G) .$$

With this multiplication, $S \times G$ becomes a discrete commutative semigroup if both S and G are. We can thus consider the l_1–algebra of $S \times G$ as in [4].

$l_1(\mathrm{S}\times\mathrm{G})$ is therefore the set of all complex-valued functions f on $\mathrm{S}\times\mathrm{G}$ which vanish except on a countable subset of $\mathrm{S}\times\mathrm{G}$ and for which

$$\|f\| = \sum_{(x,y)\in\mathrm{S}\times\mathrm{G}} |f(x,y)| \quad \text{is finite.}$$

Multiplication is defined in $l_1(\mathrm{S}\times\mathrm{G})$ by convolution as follows:

$$\text{for} \quad f, g \in l_1(\mathrm{S}\times\mathrm{G})$$

$$(f*g)(x,y) = \sum_{(u,v),(w,z),(u,v)(w,z)=(x,y)} f(u,v)\,g(w,z).$$

We note that the simple functions are dense in $l_1(\mathrm{S}\times\mathrm{G})$ [cf., 1]. We now state and prove the main result of this section.

3.3. THEOREM. *If* S *and* G *are discrete commutative semigroups, then* $l_1(\mathrm{S}, l_1(\mathrm{G}))$ *is isomorphic and isometric with* $l_1(\mathrm{S}\times\mathrm{G})$.

Proof. Let $\mathfrak{D}$ be the subset of $l_1(\mathrm{S}, l_1(\mathrm{G}))$ as defined in 3.1, then $\mathfrak{D}$ is a dense subspace of $l_1(\mathrm{S}, l_1(\mathrm{G}))$. Let $f\in\mathfrak{D}$ with $f=\sum_{i=1}^{n} h_i k_i$ where $h_i \in l_1(\mathrm{S})$ and $k_i \in l_i(\mathrm{G})$ $i = 1, 2, \cdots, n$. Define a function f_φ on $\mathrm{S}\times\mathrm{G}$ as follows: for $(x,y)\in\mathrm{S}\times\mathrm{G}$;

$$f_\varphi(x,y) = \sum_{i=1}^{n} h_i(x)\,k_i(y). \tag{3.3.1}$$

Using Theorem 12–44 of [2], it can be shown that $\sum_{(x,y)\in\mathrm{S}\times\mathrm{G}} |f_\varphi(x,y)|$ is finite.

Consequently $f_\varphi \in l_1(\mathrm{S}\times\mathrm{G})$. Denote by Π the set of all functions $f_\varphi \in l_1(\mathrm{S}\times\mathrm{G})$. Then $\Pi \subset l_1(\mathrm{S}\times\mathrm{G})$. Consider a mapping T on $\mathfrak{D}$ to Π defined as follows: for $f\in\mathfrak{D}$,

$$\mathrm{T}f = f_\varphi.$$

For $f, g\in\mathfrak{D}$, let $f=\sum_{i=1}^{n} h_i k_i$ and $g=\sum_{j=1}^{n} c_j d_j$, then

$$f*g = \sum_{i=1}^{n}\sum_{j=1}^{n} (h_i * c_j)(k_i * d_j)$$

Hence $\mathrm{T}(f*g) = (f*g)_\varphi$, where $(f*g)_\varphi$ is defined by

$$(f*g)_\varphi(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} (h_i * c_j)(x)\,(k_i * d_j)(y). \tag{3.3.2}$$

Consider $(f_\varphi * g_\varphi)(x,y) = \sum_{(u,v),(w,z),(u,v)(w,z)=(x,y)} f_\varphi(u,v)\,g_\varphi(w,z).$

By (3.3.1) $$(f_\varphi * g_\varphi)(x, y) = \sum_{(uw, vz)=(x,y)} \left[\sum_{i=1}^{n} h_i(u)\, k_i(v) \sum_{j=1}^{m} c_j(w)\, d_j(z)\right]$$

$$= \sum_{(uw, vz)=(x,y) \in S\times G} \left[\sum_{i=1}^{n} \sum_{j=1}^{m} h_i(u)\, k_i(v)\, c_j(w)\, d_j(z)\right],$$

Using Theorem 12–42 of [2],

$$(f_\varphi * g_\varphi)(x, y) = \sum_{uw=x} \sum_{vz=y} \left[\sum_{i=1}^{n} \sum_{j=1}^{m} h_i(u)\, k_i(v)\, c_j(w)\, d_j(z)\right].$$

$$\sum_{uw=x} \sum_{vz=y} \sum_{i=1}^{n} \sum_{j=1}^{m} h_i(u)\, k_i(v)\, c_j(w)\, d_j(z)$$

is absolutely convergent and hence we can rearrange the series without changing its sum. Consequently,

$$(f_\varphi * g_\varphi)(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} (h_i * c_j)(x)\,(k_i * d_j)(y).$$

Thus $f_\varphi * g_\varphi = (f * g)_\varphi$ and $T(f * g) = Tf * Tg$.

So that T is multiplicative. A straight-forward calculation shows that T is well-defined and 1–1. Clearly T is onto and linear. T is thus an isomorphism of $\mathfrak{D}$ onto $\Pi \subset l_1(S \times G)$. Furthermore

$$\|Tf\| = \|f_\varphi\| = \sum_{(x,y)\in S\times G} |f_\varphi(x, y)| = \sum_{x\in S} \sum_{y \in G} \left|\sum_{i=1}^{n} h_i(x)\, k_i(y)\right| =$$

$$= \sum_{x\in S} \left\|\sum_{i=1}^{n} (h_i(\cdot)\, k_i)(x)\right\|_{l_1(G)} = \left\|\sum_{i=1}^{n} h_i\, k_i\right\|_{l_1(S, l_1(G))},$$

where

$$f = \sum_{i=1}^{n} h_i\, k_i .$$

The set of all functions ψ_φ defined by

$$\psi_\varphi(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij}\, \xi_{E_i}(x)\, \xi_{E_j}(y)$$

are simple functions in $l_1(S \times G)$ and are contained in Π. It follows that Π is a dense subspace of $l_1(S, l_1(G))$. Thus T is an isometric isomorphism of a dense subspace $\mathfrak{D}$ onto a dense subspace Π. Taking closures with respect to their corresponding norm topologies, it follows that T is an isometric isomorphism of $l_1(S, l_1(G))$ onto $l_1(S \times G)$ and the proof is complete.

3.4. COROLLARY. *If* S *and* G *are discrete commutative semigroups, then* $\widehat{\overline{S \times G}}$ *is homeomorphic with* $\hat{S} \times \hat{G}$.

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