Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Bruce Calvert

Harnack's Theorems on convergence for non linear operators

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **52** (1972), n.5, p. 622–630.

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<http://www.bdim.eu/item?id=RLINA_1972_8_52_5_622_0>

Matematica. — *Harnack's Theorems on convergence for nonlinear operators*. Nota di BRUCE CALVERT, presentata (*) dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — In questa Nota si studiano i Teoremi di Harnack per equazioni non-lineari alle derivate parziali di tipo ellittico.

INTRODUCTION

This paper shows that Harnack's Theorems on convergence hold in a nonlinear setting. Harnack's first and second convergence Theorems (Kellog [1], pages 248, 263) are as follows.

FIRST THEOREM

Suppose o is a bounded open subset of R^n, and u_n is a sequence of continuous real valued functions on the closure of o. Suppose $\Delta u_n = 0$ in o, (where $\Delta u = \operatorname{div}(u_x)$) and u_n converges uniformly on the boundary of o. Then u_n converges uniformly on o (i.e. in L^∞) and its limit u satisfies $\Delta u = 0$ in o.

SECOND THEOREM

Suppose o is a bounded open subset of R^n, and u_n is a sequence of continuous real valued functions on o. Suppose $\Delta u_n = 0$ in o, u_n is increasing, and $u_n(y)$ is bounded for some y in o. Then u_n converges in L^∞_{loc}, and its limit u satisfies $\Delta u = 0$ in o.

Both these Theorems hold when Δ is replaced by A defined by $Au = -\operatorname{div}(|u_x|^{m-2} u_x)$ for some m in $(1, \infty)$. The aim of this note is to show that these theorems hold for a class of operators A which are defined by:

$$(Au, v) = \int (a(x, u, u_x), v_x) + b(x, v, v_x)\, v \tag{1}$$

for u in $W^{1,m}_{\text{loc}}$ and v in $W^{1,m}$ with compact support. The structure of (1) will be governed by the following conditions. o is a bounded open subset of R^n and m is in $(1, \infty)$. Conditions (2) and (3) guarantee that (1) is defined for u and v as given.

(2) *a*) $o \times R \times R^n \to R^n$ and *b*) $o \times R \times R^n \to R$ are measurable in x for (z, p) in $R \times R^n$ and continuous in (z, p) for almost all x in o.

$$|a(x, z, p)| + |b(x, z, p)| \le c(|z|^{m-1} + |p|^{m-1} + 1) \tag{3}$$

(*) Nella seduta del 13 maggio 1972.

(4) $(a(x,z,p)-a(x,y,q),p-q)+(b(x,z,p)-b(x,y,q))(z-y)\geq 0$

(4 *a*) $cl(\mathrm{o})$ is the union of the closures of a sequence of open sets o_i which have boundaries of measure 0. For each set o_i there is s_i in R^n, s'_i in R, one of them nonzero, such that for almost all x in o_i if we have equality in (4) then

$$(p-q,s_i)+(z-y,s'_i)=0\,.$$

For almost all x_i in o_i there is a finite set $j,k,\cdots,m$ such that x_i may be joined to a point x_j in o_j by a line l_i in the direction s_i; x_j to a point x_k in o_k by a line l_j in the direction s_j; $\cdots$; and x_m may be joined to a point in the complement of $cl(\mathrm{o})$ by a line l_m in the direction S_m. (More generally we may let s_i be a smooth nonzero divergence free vector field and replace l_i by the integral curve of s_i, and s'_i by an L^∞ function).

(5) $(a(x,z,p),p)\geq d\,|p|^m-c\,(|z|^m+1)$

(6) $b(x,u,v)-b(x,u-k,v)-\operatorname{div}(a(x,u,v)-a(x,u-k,v))\geq 0$ for $u:\mathrm{o}\to\mathrm{R}$ and $v:\mathrm{o}\to\mathrm{R}^n$ in L^m and $k\geq 0$ a constant function.

(7) Near the boundary of o,

$$(a(x,z,p)-a(x,y,q),p-q)\geq h(z,y)\min(|p|^{m-2},|q|^{m-2})\,|p-q|^2$$
$$-f(z,y)(\sup(|p|^m,|q|^m)+c)$$

where $h:\mathrm{R}\times\mathrm{R}\to\{r \text{ in } \mathrm{R}:r>0\}$ is continuous and $f:\mathrm{R}\times\mathrm{R}\to\{r \text{ in } \mathrm{R}:r\geq 0\}$ is continuous and satisfies $f(z,y)\to 0$ as $z-y\to 0$

(8) Near the boundary of o,

$$a(x,z,p)=d(x,p)+c(x,z),$$
$$|d(x,p)|\leq c\,|p|^{m-1},$$
$$(d(x,p),p)\geq d\,|p|^m,$$

and with $h>0$

$$(d(x,p)-d(x,q),p-q)\geq h\min(|p|^{m-2},|q|^{m-2})|p-q|^2\,.$$

(9) $$|a(x,z,p)|+|b(x,z,p)|\leq c\,(|z|^{m-1}+|p|^{m-1})$$

and

$$(a(x,z,p),p)\geq d\,|p|^m-c\,|z|^m\,.$$

The Author expresses gratitude to Prof. Stampacchia for his interest and advice.

The Convergence Theoremes. In this section, we state a Theorem, and divide its proof into seven lemmata.

THEOREM. *Harnack's first Theorem holds for A defined by* (1) *if we assume* (2), (3), (4), (4 *a*), (5), (6) *and* (7) *when* $m \leq 2$, *and* (8) *if we allow* m *to be possibly* > 2. *Harnack's second theorem holds for A defined by* (1) *if we assume* (9), (2), (4), (6) *and* (7) *when* $m \leq 2$ *and* (8) *if we allow* m *to be possibly* > 2.

LEMMA 1. *Suppose* u_n *is a sequence of real valued continuous functions on the closure of* o, *and* $Au_n = 0$ *where* A *given by* (1) *satisfies* (2), (3), (4), (4 *a*), *and* (6). *Then for all* n, m

$$\sup \{(u_n - u_m)(x) : x \quad in \ \mathrm{o}\} = \sup \{(u_n - u_m)(x) : x \quad in \ bdy\ (\mathrm{o})\},$$

that is, the weak maximum principle holds.

In particular if u_n converges uniformly on bdy (o) then u_n converges uniformly on cl (o) to a continuous function $u : \mathrm{cl}(\mathrm{o}) \to \mathrm{R}$.

Proof. Conditions (2) and (3) imply A is defined by (1). Suppose $Au = Av = 0$ on o, u and v being continuous on cl (o) and satisfying $u \leq u + k$ on bdy (o) where $k \geq 0$ is a constant. We have to show that $u \leq v + k$ on o. Given o′ compactly contained in o we may, since u and v are continuous, take o″ compactly contained in o such that $\mathrm{o}' \subset \mathrm{o}''$ and $u \leq v + k$ on bdy (o″) in the sense of $W^{1,m}(\mathrm{o}'')$. Since $Au = Av = 0$ on o″, and $(u - v - k)^+$ is in $W_0^{1,m}(\mathrm{o}'')$,

$$\begin{aligned} 0 &= (Au - Av, (u - v - k)^+) \\ &= \int (a(x, u - k, (u - k)_x) - a(x, v, v_x), ((u - k - v)^+)_x) \\ &+ \int (b(x, u - k, (u - k)_x) - b(x, v, v_x))(u - k - v)^+ \\ &+ \int (a(x, u, u_x) - a(x, u - k, (u - k)_x), ((u - k - v)^+)_x) \\ &+ \int (b(x, u, u_x) - bx, u - k, (u - k)_x)(u - k - v)^+ . \end{aligned}$$

By (6) the sum of the last two terms is nonnegative.

We will show (4 *a*) implies A is strictly T–monotone, that is if $(u - v)^+$ is in $W_0^{1,m}(\mathrm{o}'')$ and

$$(Au - Av, (u - v)^+) = 0$$

then $(u - v)^+ = 0$. This gives $((u - k) - v)^+ = 0$. Hence, $u \leq v + k$ on o″ and consequently on o by the arbitrary choice of o′.

To prove A is strictly T–monotone we may assume by induction that $(Au - Av, (u - v)^+) = 0$, $(u - v)^+ = 0$ on o_j, and show $(u - v)^+ = 0$ on o_i.

Since

$$\begin{aligned} 0 = \int (a(x, u, u_x) - a(x, v, v_x), (u - v)_x^+) + \\ (b(x, u, u_x) - b(x, v, v_x))(u - v)^+ , \end{aligned}$$

we have by (4 *a*) that a.e. in o_i,

$$((u-v)_x^+ , s) + (u-v)^+ s' = 0.$$

If $s = 0$ then $s' \neq 0$ and hence $(u-v)^+ = 0$. Otherwise we may suppose $s = (1 , 0 \cdots 0)$, giving

$$(u-v)_{x_1}^+ + (u-v)^+ s' = 0 .$$

It follows that if

$$w_z = (e^{s'x_1}(u-v)^+) \qquad \text{then} \quad w_{x_1} = 0 .$$

Since w is in $w^{1,m}$, we have almost everywhere (where x is joined to x_j in o_j by a line in the direction s, contained in o_i and o_j except for a set of measure zero)

$$w(x) = \int w_{x_1} \, dx_1 = 0 ,$$

the integral being along the line from x_j to x. Hence $(u-v)^+ = 0$ a.e. on o_i.

LEMMA 2. *Suppose u_n a sequence of real valued functions on* o, *and* $\mathrm{A}u_n = 0$ *where* A *given by* (1) *satisfies* (2), (3) *and* (5).

Then u_n is bounded in $\mathrm{W}^{1,m}_{\mathrm{loc}}(\mathrm{o})$ *if it is bounded in* $\mathrm{L}^m_{\mathrm{loc}}(\mathrm{o})$, *and in particular if it is bounded in* $\mathrm{L}^\infty(\mathrm{o})$.

Proof. Take y in $\mathrm{C}_0^\infty(\mathrm{o})$, $y \geq 0$, and suppose $\mathrm{A}u = 0$, A being defined by (1) since (2) and (3) hold.

$$\begin{aligned} 0 &= (\mathrm{A}u , uy^m) \\ &= \int (a(x , u , u_x) u_x) y^m + (a(x , u , u_x) , y_x) u \, my^{m-1} + b(x , u , u_x) uy^m . \end{aligned}$$

By (3) and (5)

$$\begin{aligned} \int d |u_x|^m y^m \leq \int c(|u|^m + 1) y^m &+ c(|u|^{m-1} + |u_x|^{m-1} + 1) |y_x| |u| my^{m-1} \\ &+ c(|u|^{m-1} + |u_x|^{m-1} + 1) |u| y^m . \end{aligned}$$

We absorb the terms in $|u_x|$ into the left hand side by Young's inequality

$$ab \leq p^{-1}(ea)^p + q^{-1}(b/e)^q$$

where $e > 0$, $p > 1$, $p^{-1} + q^{-1} = 1$.

The result is

$$\int |u_x|^m y^m \leq \int c(c , d , m)(y^m + |y_x|^m)(1 + |u|^m).$$

LEMMA 3. *Let* A *be given by* (1), *satisfying* (2) *and* (3) *and* (4). *Then u in* $\mathrm{W}^{1,m}$ *is a solution of* $\mathrm{A}u = 0$ *if and only if*

$$(\mathrm{A}v , v - u) \geq 0$$

for all v in $\mathrm{W}^{1,m}_{\mathrm{loc}}$ *such that $v = u$ on a neighbourhood of bdy* (o).

Proof. Suppose $Au = 0$. By (4), if v is in $W^{1,m}_{loc}$ and $v - u$ is in $W^{1,m}$ with compact support,

$$(Av, v - u) \geq (Au, v - u) = 0 .$$

Conversely, suppose that for v in $W^{1,m}$, $v = u$ on a neighbourhood of bdy (o), and

$$(Av, v - u) \geq 0.$$

Take w in $W^{1,m}$ with compact support, t in $(0, 1)$, and set $v_t = u + tw$. Since $(Av_t, v_t - u) \geq 0$; if follows that

$$(Av_t, w) \geq 0 .$$

Since v_t converges to u and A is continuous by (2) and (3) we have

$$(Au, w) \geq 0 ,$$

which implies $Au = 0$.

LEMMA 4. *Suppose* (2), (3) *and* (4) *hold, and* A *is given by* (1). *Let* u_n *be a sequence of functions satisfying* $Au_n = 0$. *Let* u *be in* $W^{1,m}_{loc}(o)$ *and let* o'' *be a neighbourhood of bdy* (o). *Suppose* u_n *converges to* u *in* $W^{1,m}_{loc}(o'')$, *and* u_n *converges weakly to* u *in* $W^{1,m}(o')$ *for* o' *compactly contained in* o. *Then* $Au = 0$.

Proof. Take v in $W^{1,m}$ such that $v = u$ on a neighbourhood of bdy (o). Take an open set o' compactly contained in o which contains the support of $v - u$ and the complement of o''. Let $f : o \to [0, 1]$ be smooth, its support contained in o'', and equal to 1 on a neighbourhood of the boundary of o'. Let $v_n = v + (u_n - u) f$.

By Lemma 3, since $Au_n = 0$ and $v_n = u_n$ on a neighbourhood of bdy (o'),

$$(Av_n, v_n - u_n) \geq 0 .$$

Since u_n converges weakly to u in $W^{1,m}_{loc}(o')$, and v_n converges to v in $W^{1,m}(o')$ we have

$$(Av, v - u) \geq 0 .$$

By Lemma 3 again, $Au = 0$ on o', and hence on o.

LEMMA 5. *Suppose* $m \leq 2$. *Suppose* (2), (3) *and* (7) *hold in a neighbourhood* o'' *of bdy* (o).

Let u_n *be a sequence of real valued functions satisfying* $Au_n = 0$, A *given by* (1). *Suppose* u_n *is convergent in* $L^\infty(o'')$, *and bounded in* $W^{1,m}_{loc}(o'')$. *Then* u_n *is convergent in* $W^{1,m}_{loc}(o'')$.

Proof. Let $y \geq 0$ be in $C_0^\infty(o'')$. By Holder's inequality (here we need $m \leq 2$) we have:

$$\int |u_x - v_x|^m y^2 \leq \left(\int (\sup(|u_x|, |v_x|))^m y^2 \right)^{(2-m)/2} \cdot \tag{10}$$

$$\left(\int (\sup(|u_x|, |v_x|))^{m-2} |u_x - v_x|^2 y^2 \right)^{m/2} .$$

Since u_n is bounded in $W^{1,m}_{\mathrm{loc}}(o'')$, the first integral on the right hand side is bounded. Since u_n is bounded in $L^{\infty}_{\mathrm{loc}}(o'')$ there is $h > 0$ such that $h(u, v) \geq \geq h > 0$ on the support of y. Then by (7) we have

$$\begin{aligned}(11)\qquad &\int (\sup(|u_x|, |v_x|))^{m-2} |u_x - v_x|^2 y^2 \\ &\leq h \int (a(x, u, u_x) - a(x, v, v_x), u_x - v_x) y^2 \\ &+ \int f(u, v)(\sup(|u_x|^m, |v_x|^m) + c) y^2 .\end{aligned}$$

Since u_n is convergent in L^{∞}, the second integral converges to 0. By (11) if $m = 2$ or (10) and (11) if $m < 2$, we have to show that the first integral on the right hand side of (11) converges to 0.

Since $(Au - Av, (u - v) y^2) = 0$, we have

$$\begin{aligned}(12)\qquad &\left| \int (a(x, u, u_x) - a(x, v, v_x), u_x - v_x) y^2 \right| \\ &\leq \int |a(x, u, u_x) - a(x, v, v_x)| |y_x| 2y |u - v| \\ &+ \int |b(x, u, u_x) - b(x, v, v_x)| y^2 |u - v| .\end{aligned}$$

Using (1) and the convergence of u_n in L^{∞}, the right hand side of (12) converges to 0.

LEMMA 6. *Suppose that m is not necessarily ≤ 2. Suppose (2) and (3) hold, and A is given by (1). Suppose that u_n is a sequence of real valued functions satisfying $Au_n = 0$. Suppose u_n is convergent in $L^{\infty}_{\mathrm{loc}}(o'')$ and bounded in $W^{1,m}_{\mathrm{loc}}(o'')$ where o'' is a neighbourhood of bdy (o) on which (8) holds. Then u_n is convergent in $W^{1,m}_{\mathrm{loc}}(o'')$.*

Proof. Let y, u, v be as in Lemma 5. Let F be the set where $|u_x| < e |v_x|$, $e > 0$ to be chosen. By (8)

$$\begin{aligned}\int_F |v_x|^m y^2 &\leq \mathrm{d}^{-1} \int_F (\mathrm{d}(x, v_x), v_x) y^2 \\ &\leq \mathrm{d}^{-1} \int_F (\mathrm{d}(x, v_x) - \mathrm{d}(x, u_x), v_x - u_x) y^2 \\ &+ \mathrm{d}^{-1} \int_F c |u_x|^{m-1} (|v_x| + |u_x|) y^2 \\ &+ \mathrm{d}^{-1} \int_F c |v_x|^{m-1} |u_x| y^2.\end{aligned}$$

Taking e small we have the existence of C depending on c, d such that

$$\int_F |v_x|^m y^2 \leq C \int_F (\mathrm{d}(x, v_x) - \mathrm{d}(x, u_x), v_x - u_x) y^2 .$$

It follows that there is another constant C such that

$$(13) \qquad \int_F |u_x - v_x|^m y^2 \leq C \int_F (\mathrm{d}(x, u_x) - \mathrm{d}(x, v_x), u_x - v_x) y^2.$$

Similarly if G is the set where $|v_x| < e |u_x|$ then

$$(14) \qquad \int_G |u_x - v_x|^m y^2 \leq C \int_G (\mathrm{d}(x, u_x) - \mathrm{d}(x, v_x), u_x - v_x) y^2.$$

Let E be the complement of F and G, that is the set where $|u_x| \neq 0$, $|v_x| \neq 0$, and

$$(15) \qquad e |v_x| \leq |u_x| \leq e^{-1} |v_x| .$$

We take m^{th} powers in (8) to give on E

$$|u_x - v_x|^m \leq C (\sup (|u_x|^{2-m}, |v_x|^{2-m}))^{m/2} (\mathrm{d}(x, u_x) - \mathrm{d}(x, v_x), u_x - v_x)^{m/2}.$$

By Cauchy's inequality this gives

$$(16) \qquad \int_E |u_x - v_x|^m y^2 \leq C \left(\int_E (\mathrm{d}(x, u_x) - \mathrm{d}(x, v_x), u_x - v_x) y^2 \right)^{1/2} .$$

$$\left(\int_E (\sup (|u_x|^{2-m}, |v_x|^{2-m}))^m (\mathrm{d}(x, u_x) - \mathrm{d}(x, v_x), u_x - v_x)^{m-1} y^2 \right)^{1/2} .$$

We obtain a bound for the second integral on the right hand side of (16) by using (8) and (15). We now show the first integral on the right converges to zero.

$$(17) \quad \int_E (\mathrm{d}(x, u_x) - \mathrm{d}(x, v_x), u_x - v_x) y^2 \leq \int_0 (\mathrm{d}(x, u_x) - \mathrm{d}(x, v_x), u_x - v_x) y^2$$

$$\leq \left| \int (a(x, u, u_x) - a(x, v, v_x), u_x - v_x) y^2 \right| + \int |c(x, u) - c(x, v)| (u_x - v_x) y^2.$$

The first integral on the right converges to 0 by (12). The second converges to 0 because of convergence of u_n in L^∞_{loc}.

Since (17) converges to zero, so does (16), and similarly (13) and (14). Hence, u_n converges in $W^{1,m}_{\text{loc}}(0'')$.

LEMMA 7. *Suppose* (2), (6) *and* (9) *hold and* A *is given by* (1). *Suppose* M *is a real number*, $M \geq 0$, *and* u_n *is an increasing sequence of functions satisfying* $Au_n = 0$ *and* $u_n \geq -M$ *on* O. *Suppose there is a point* y *in* O *such that* $u_n(y)$ *is bounded. Then* u_n *is convergent in* L^∞_{loc}.

Proof. We shall refer to Trudinger [4] for much of this proof. We may assume $M \geq -2u_n$. Let u be an element of the sequence u_n. Let $y \geq 0$ be in $C_0^\infty(K)$, K being a cube in O. Let p be > 0.

By Serrin [3] u is in $L^\infty(K)$. It then follows that $y^m(u+M)^p$ is in $W_0^{1,m}$. $(Au, y^m(u+M)^p) = 0$ gives

$$0 = \int (a(x, u, u_x), u_x)\, p(u+M)^{p-1} y^m + \int (a(x, u, u_x)\, y_x)\, my^{m-1}(u+M)^p + \\ + \int b(x, u, u_x)\, y^m (u+M)^p.$$

By (9), this implies

$$\int |u_x|^m (u+M)^{p-1} y^m \leq c\mathrm{d}^{-1} \int |u|^m (u+M)^{p-1} y^m \\ + \mathrm{d}^{-1} p^{-1} \int c(|u|^{m-1} + |u_x|^{m-1})\, my^{m-1}(u+M)^p \\ + \mathrm{d}^{-1} p^{-1} \int c(|u|^{m-1} + |u_x|^{m-1})\, y^m (u+M)^p.$$

By Young's inequality we take the terms in $|u_x|$ to the left hand side, and use the fact $u + M \geq u$, to give

$$(18) \qquad \int |u_x|^m (u+M)^{p-1} y^m \leq C(c, \mathrm{d})\,(1+p^{-1})^m \int (y^m + |y_x|^m)\,(u+M)^{m+p-1}.$$

Letting $w = u + M$ (18) becomes

$$\int |w_x|^m w^{p-1} y^m \leq C(1+p^{-1})^m \int (y^m + |y_x|^m)\, w^{m+p-1},$$

which is the formula (1.20) of [4].

The conclusion of Theorem 1.3 of [4], follows for w by repeating the Proof of this Theorem from (1.20) onward. That is, for $\gamma > m - 1$

$$(19) \qquad \max_{K(\rho)} w \leq c\rho^{-n/\gamma} \|w\|_{\gamma, K(2\rho)}.$$

Since (6) implies that $w = u + M$ is a positive supersolution, by [4, Theorem 1.2] for $\gamma < n(m-1)(n-m)^{-1}$ we obtain

$$(20) \qquad \rho^{-n/\gamma} \|w\|_{\gamma, K(2\rho)} \leq c \min_{K(\rho)} w.$$

We note $u_n(y)$ is defined since the u_n are continuous by Ladyzhenskaya and Ural'tseva [2, Theorem 1.1, page 251]. Since $u_n(y)$ are bounded so is min (u_n) over o' containing y. If o' is compactly contained in o then (19) and (20) imply the existence of C such that

$$\max_{0'} (u_n + M) \leq C \min_{0'} (u_n + M).$$

Since u_n is bounded in $L^\infty(o)$, by [2, Theorem 1.1, page 251] u_n is bounded in $C_{0'\alpha}\ cl(o')$ for o' compactly contained in o. Consequently u_n is compact in the space of continuous functions on $cl(o')$, and since it is an increasing sequence it converges uniformly on $cl(o')$.

References

[1] O. D. Kellog, *Foundations of Potential theory*, Springer Verlag, Berlin 1929.

[2] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Lineynyye Kvazilineynyye Uravneniya Ellipticheskogo Tipa*, Nauka Press, Moscow 1964, English translation Linear and Quasilinear Elliptic Equations, Academic Press, New York 1968.

[3] J. Serrin, *Local behaviour of solutions of quasilinear equations*, «Acta Math», *111*, 302–347 (1964).

[4] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, «Comm. Pure Appl. Math.», *20*, 721–747 (1967).