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On the exceptional waves in relativistic magnetohydrodynamics (MHDR)

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Fisica matematica. — *On the exceptional waves in relativistic magnetohydrodynamics (MHDR)* (*). Nota di ANTONIO GRECO, presentata (**) dal Corrisp. C. CATTANEO.

RIASSUNTO. — Si dimostra che in un fluido perfetto relativistico carico di conducibilità infinita tutte le onde sono eccezionali, nel senso che non evolvono in onde d'urto, solamente se il fluido è incomprimibile.

1. INTRODUCTION

It is well known that when a field satisfies a non-linear hyperbolic system, reduced to a first order quasi-linear system [1], [2], the corresponding wave fronts (i.e. the characteristics) are accelerated and sooner or later they turn into shocks, in the sense that the discontinuities of the first order derivatives of the field across the characteristics go to infinity after an evaluable time interval. However Lax [3], for non-linear fields depending on t and x only, has pointed out that there exist some waves for which this phenomenon does not arise. Lax has also given the conditions under which this happens and called such waves "exceptional". The generalized n–dimensional case has been worked out by Boillat [4], which recently [5] has expressed in covariant form the exceptionality criterion and, in particular, he proved that, if characteristic equation is given by

$$(a) \qquad G^{\alpha\beta\cdots\rho}\,\varphi_\alpha\,\varphi_\beta\cdots\varphi_\rho = 0 \qquad \left(\varphi_\alpha \equiv \frac{\partial\varphi}{\partial x^\alpha}\right),$$

which is covariant if the field equations are covariant, the solutions $\varphi(x^\alpha)$ of (a) give a family of exceptional waves $\varphi(x^\alpha) = \text{const.}$, provided that

$$(b) \qquad \varphi_\alpha\,\varphi_\beta\cdots\varphi_\rho\,\delta G^{\alpha\beta\cdots\rho} = 0$$

when (a) is satisfied. In (b) and in the following $\delta \equiv \left[\frac{\partial}{\partial\varphi}\right]$ indicates the discontinuity operator across $\varphi(x^\alpha) = \text{const.}$ If all the waves connected with the system are exceptional, the field is called completely exceptional.

In the present work we will show, in the framework of the above mentioned theory, that MHDR system is completely exceptional only if the charged fluid is incompressible.

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(**) Nella seduta dell'8 aprile 1972.

In Sec. II the MHDR equations will be recalled together with some consequences. In Sec. III equations for the discontinuities of the first order derivatives across wave surfaces, and characteristic equations of material waves, Alfvén waves and hydrodynamical (slow and fast) waves respectively will be given. In the last section proof will be given that for the first two types of waves the exceptionality criterion is always satisfied, while the hydrodynamical waves are exceptional only in the case of incompressible fluids.

2. Field equations

Let us consider a perfect charged relativistic thermodynamical fluid with constant magnetic permeability μ and infinite conductivity. By following Lichnerowicz [6], the field equations can be written [1]:

$$\left\{\begin{array}{l} \nabla_\alpha \mathrm{T}^{\alpha\beta} = 0 \\ \nabla_\alpha (u^\alpha b^\beta - u^\beta b^\alpha) = 0 \\ \nabla_\alpha (r u^\alpha) = 0 \, , \end{array}\right. \tag{1}$$

where $\mathrm{T}^{\alpha\beta}$ is the energy tensor:

$$\mathrm{T}^{\alpha\beta} = (c^2 rf + |b|^2)\, u^\alpha u^\beta - \left(p + \frac{1}{2}|b|^2\right) g^{\alpha\beta} - b^\alpha b^\beta ,$$

c being the velocity of light, r the proper material density (number of particles), $f = 1 + i/c^2$ the index of the fluid, i the specific entalpy, p the pressure, u^α the unitary 4–velocity ($u^\alpha u_\alpha = 1$) and $|b|^2 = -b^\alpha b_\alpha$ with $b^\alpha = \sqrt{\mu}\, \overset{*}{\mathrm{F}}{}^{\alpha\beta} u_\beta$ being $\overset{*}{\mathrm{F}}{}^{\alpha\beta} = \frac{1}{2} \mathrm{E}^{\alpha\beta\mu\nu} \mathrm{F}_{\mu\nu}$ the dual of the electromagnetic tensor $\mathrm{F}_{\mu\nu}$; of course b^α is a space-like vector ($b^\alpha u_\alpha = 0$). Moreover, the relation

$$c^2 \, \mathrm{d}f = \frac{1}{r} \, \mathrm{d}p + \mathrm{T} \, \mathrm{dS} \, , \tag{2}$$

which came from the thermodynamical principles, is assumed to hold. Here T and S are the proper temperature of the fluid and its specific proper entropy respectively. In the following we will assume p and S as the thermodynamical independent variables.

From eqs. (1), taking into account eq. (2), it is possible to obtain the useful equations

$$u^\alpha \partial_\alpha \mathrm{S} = 0 \tag{3}$$

$$c^2 rf \nabla_\alpha b^\alpha + b^\alpha \partial_\alpha p = 0 \, . \qquad \left(\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}\right) \tag{4}$$

(1) ∇_α is the operator of covariant derivation with respect to a riemannian hyperbolic metric $(+ - - -)$ which can be expressed in local coordinates in the usual form $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$.

3. DISCONTINUITIES AND CHARACTERISTIC EQUATIONS

The equations for the discontinuities of first order derivatives across the wave fronts $\varphi(x^\alpha) = \text{const.}$ are simply obtained by making the following replacement in the field equations

$$\nabla_\alpha\,, \partial_\alpha \to \varphi_\alpha\,\delta\,.$$

On the other hand, both the discontinuities and characteristic equations have been elsewhere extensively discussed [6], [7], so that we give here only a brief outline of the derivation. By interchanging operators as above indicated, from eqs. (1) follows

$$\begin{aligned} &\mathrm{U}\left[c^2 r \delta(f u^\alpha) + \delta(|b|^2 u^\alpha)\right] + |b|^2 u^\alpha \varphi_\beta\, \delta u^\beta + \\ &+ g^{\alpha\beta}\varphi_\beta\,\delta\left(p + \frac{1}{2}|b|^2\right) - \mathrm{B}\,\delta b^\alpha - b^\alpha \varphi_\beta\,\delta b^\beta = 0 \end{aligned} \tag{5}$$

$$\mathrm{U}\,\delta b^\alpha + b^\alpha \varphi_\beta\,\delta u^\beta - \mathrm{B}\,\delta u^\alpha - u^\alpha \varphi_\beta\,\delta b^\beta = 0 \tag{6}$$

$$\mathrm{U}\,\delta r + r\varphi_\beta\,\delta u^\beta = 0\,, \tag{7}$$

in which $\mathrm{U} = u^\alpha \varphi_\alpha$, $\mathrm{B} = b^\alpha \varphi_\alpha$ and use has been made of the fact that the gravitational potentials are continuous to the first order [8].

Moreover, from eqs. (3) and (4) it follows that

$$\mathrm{U}\,\delta \mathrm{S} = 0 \tag{8}$$

$$c^2 r f \varphi_\alpha\,\delta b^\alpha + \mathrm{B}\,\delta p = 0\,. \tag{9}$$

It is possible to show, from eqs. (5)–(9), that discontinuities of the first order derivatives could take place across the surfaces $\varphi(x^\alpha) = \text{const.}$ for which $\varphi(x^\alpha)$ is solution of any of the following equations

$$\mathrm{N}_1 = \mathrm{U} = 0 \to \text{material waves}$$

$$\mathrm{N}_2 = (c^2 r f + |b|^2)\,\mathrm{U}^2 - \mathrm{B}^2 = 0 \to \text{Alfvén waves}$$

$$\mathrm{N}_4 = c^2 r f(\gamma - 1)\mathrm{U}^4 + (c^2 r f + \gamma\,|b|^2)\mathrm{U}^2\mathrm{G} - \mathrm{B}^2\mathrm{G} = 0 \to \text{hydrodynamical waves,}$$

and it will be assumed that the above three equations are mutually exclusive. In N_4 we have put $\mathrm{G} = g^{\alpha\beta}\varphi_\alpha\varphi_\beta$ and $\gamma = c^2 f r'_p$, where the prime denotes partial differentiation with respect to the subscripted variable.

It is well known that first-order discontinuities for the entropy can take place across material waves; across Alfvén waves such discontinuities can arise for tangential components of the 4–velocity and of the magnetic field, whereas across hydrodynamical waves for the normal components of the 4–velocity and of the magnetic field and for the pressure.

In the present work we are rather interested in the inverse implication:

$$N_4 \neq 0 \Rightarrow \varphi_\alpha \, \delta u^\alpha = \varphi_\alpha \, \delta b^\alpha = \delta p = 0 . \tag{10}$$

We are now in a position to study the various types of waves from the exceptionality point of view.

4. Exceptionality conditions

Let us consider first the material waves. In this case we have $N_1 = U = 0$. According to the criterion mentioned in the first section, they are exceptional if $\varphi_\alpha \, \delta u^\alpha = 0$, which is obviously verified by taking into account eq. (7).

As far as the Alfvén waves are concerned, since they are solutions of equation $N_2 = 0$, then they are exceptional if

$$U^2 \, \delta (c^2 rf + |b|^2) + 2 U (c^2 rf + |b|^2) \, \varphi_\alpha \, \delta u^\alpha - 2 B \varphi_\alpha \, \delta b^\alpha = 0 . \tag{11}$$

In order to demonstrate eq. (11) it should be observed that we assumed p and S as independent thermodynamical variables, so that

$$\delta (rf) = (rf)'_p \, \delta p + (rf)'_S \, \delta S ,$$

while, being $|b|^2 = - b^\alpha \, b_\alpha$, we have

$$\delta \, |b|^2 = - 2 \, b_\alpha \, \delta b^\alpha .$$

On the other hand, contracting eq. (6), first with u_α and then with b_α, by taking into account the relations $u^\alpha \, u_\alpha = 1$, $u^\alpha \, b_\alpha = 0$, one obtains the following equations

$$U b_\alpha \, \delta b^\alpha - |b|^2 \, \varphi_\alpha \, \delta u^\alpha + B u_\alpha \, \delta b^\alpha = 0 \tag{12}$$

$$U u_\alpha \, \delta b^\alpha = \varphi_\alpha \, \delta b^\alpha \tag{13}$$

which hold in general.

Now $N_2 = 0$, so that $N_4 \neq 0$ and $N_1 = U \neq 0$; the former inequality implies that condition (10) holds, while the latter, with the aid of eq. (8), leads to $\delta S = 0$, and eq. (11) is satisfied for the above arguments.

Therefore both material waves and Alfvén waves are exceptional, without any restrictive hypothesis on the state equation of the fluid. We recall that this fact is also true in the non-relativistic case.

Let us discuss finally the hydrodynamical waves. They are exceptional if from $N_4 = 0$ follows

$$U^4 \, \delta [c^2 rf (\gamma - 1)] + 4 U^3 c^2 rf (\gamma - 1) \, \varphi_\alpha \, \delta u^\alpha + U^2 G \delta (c^2 rf + \gamma \, |b|^2) + \\ + 2 U G (c^2 rf + \gamma \, |b|^2) \, \varphi_\alpha \, \delta u^\alpha - 2 BG \, \varphi_\alpha \, \delta b^\alpha = 0 . \tag{14}$$

We now transform eq. (14) in order to express the discontinuities in terms of δp only. Since $N_4 = 0$ we have $N_1 = U \neq 0$ which implies $\delta S = 0$, so that $\delta r = r'_p \, \delta p$, $\delta\gamma = \gamma'_p \, \delta p$ and then

$$\delta (c^2 rf) = (\gamma + 1) \, \delta p \, . \tag{15}$$

Furthermore from eq. (7), in this case, we have:

$$\varphi_\alpha \, \delta u^\alpha = - U \frac{r'_p}{r} \, \delta p \, , \tag{16}$$

and from eq. (9) we obtain:

$$\varphi_\alpha \, \delta b^\alpha = - \frac{B}{c^2 rf} \, \delta p \, . \tag{17}$$

Finally we attempt to express $\delta \, | b |^2 = - 2 \, b^\alpha \, \delta b_\alpha$ in terms of δp. Multiplying eq. (12) by U and taking into account eqs. (13) and (16) we obtain:

$$U^2 \, \delta \, | b |^2 = 2 \left(U^2 \, | b |^2 \frac{r'_p}{r} - \frac{B^2}{c^2 rf} \right) \delta p \, . \tag{18}$$

Inserting eqs. (15)–(18) into eq. (14) it follows

$$\left\{ U^4 (c^2 rf \gamma'_p - 3 \gamma^2 + 4 \gamma - 1) + U^2 G \left(| b |^2 \gamma'_p - \gamma + 1 \right) - 2 B^2 G \frac{\gamma - 1}{c^2 rf} \right\} \delta p = 0 . \tag{19}$$

Substituting in eq. (19) for $B^2 G$ the expression which is obtained from the characteristic equation $N_4 = 0$, and dividing by $U^2 \neq 0$, we see that the exceptionality criterion is satisfied if

$$K_1 \, U^2 + K_2 \, G = 0 \tag{20}$$

being

$$K_1 = c^2 rf \gamma'_p + (\gamma - 1)(3 - 5 \gamma)$$

$$K_2 = | b |^2 \left[\gamma'_p - (\gamma - 1) \frac{2 \gamma}{c^2 rf} \right] - 3 (\gamma - 1) .$$

Then the hydrodynamical waves are exceptional if, $\forall \, | b |^2$, $K_1 = K_2 = 0$.

The last conditions are verified if and only if

$$\gamma = c^2 f r'_p = 1 \, . \tag{21}$$

As observed by Lichnerowicz [9] eq. (21) leads to the following state equation (2)

$$rf - 2 \frac{p}{c^2} = \Psi(S) \, . \tag{22}$$

In this case the hydrodynamical waves, which coincide with the Alfvén waves, as is clearly seen from the respective characteristic equations, pro-

(2) We point out that in ref. [9] $\gamma = f r'_f / r$ is assumed instead of $\gamma = c^2 f r'_p$. It is easily verified, however, that by means of eq. (2) the two positions are equivalent.

pagate with the limit-velocity c. For this reason the fluids in which the thermodynamical state is ruled out by eq. (22) are called incompressible [9], or more precisely of minimum compressibility [10].

In conclusion we can say that the MHDR system is completely exceptional only when eq. (22) holds. This situation is different both from that of the non relativistic case, in which complete exceptionality takes place if the thermodynamical state is ruled out by an equation such as [4]

$$p = -\frac{\Psi(S)}{r^4} + \chi(S),$$

and from that of the relativistic hydrodynamics (no charged fluids), in which complete exceptionality can take place not only when eq. (22) holds, but also if the state equation is of type [11]

$$p = -\frac{\Psi(S)}{c^2 rf - p + \chi(S)} + \chi(S).$$

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