Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A characteristic ring of a Lie algebra extension. Nota I

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **52** (1972), n.4, p. 498–506.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_52_4_498_0>

Topologia algebrica. — *A characteristic ring of a Lie algebra extension* (*). Nota I di NICOLAE TELEMAN, presentata (**) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — In questo lavoro costruisco un «anello caratteristico» per estensioni corte di algebre di Lie. In particolare mostro che ad ogni fibrato principale può associarsi una estensione di algebre di Lie in tal guisa che si ritrova così sostanzialmente la costruzione geometrica dell'anello caratteristico del fibrato dovuta a S. S. Chern e A. Weil.

Mostro inoltre, fra l'altro, che l'anello caratteristico qui considerato permette di introdurre un anello caratteristico per la coomologia di I. M. Gelfand e D. B. Fuks [4].

§ 1. INTRODUCTION

It is known [6], Ch. III, § 9 that any extension ($\mathcal{E}$) of the Lie algebra T by an *abelian* Lie algebra H, $(\mathcal{E}) \equiv 0 \to H \to P \to T \to 0$, is uniquely determined, up an equivalence, by an element of the cohomology group $\mathcal{H}^2(T, H)$.

By the other hand, there is a geometric construction of the real characteristic ring of smooth principal bundles, due to S. S. Chern and A. Weil.

In this Note we construct a characteristic ring for any split Lie algebra extension ($\mathcal{E}$), with H not necessary abelian, in conditions which will be defined below, using in essence the method of S. S. Chern and A. Weil.

We prove that for any smooth principal bundle we can associate a Lie algebra extension which furnishes, with our definitions, for a particular system of coefficients, the characteristic ring of S. S. Chern and A. Weil. By using another system of coefficients, we construct for any principal bundle, a characteristic ring which is a subring of the cohomology ring studied by I. M. Gelfand and D. B. Fuks in [4].

Another application of our study is the construction of a characteristic ring for bundles of type ISO (E, F), where E, F are vector bundles.

For any split extension ($\mathcal{E}$) we define "connections", the "connection form", the "curvature", "covariant derivative", with which we prove the "structure equation", the "Bianchi identity". Our constructions of the curvature and connection form, and the corresponding classical definitions in the specific case are different, but the curvature coincides with the curvature of the associated linear connection.

Being in the possession of the curvature for any split extension ($\mathcal{E}$), we use the geometrical construction of the characteristic ring as it is presented

(*) Lavoro escguito presso l'Istituto Matematico «G. Castelnuovo» dell'Università di Roma, come professore visitatore del C.N.R.

(**) Nella seduta dell'8 aprile 1972.

in the paper by Raoul Bott and S. S. Chern [2], § 2 (we use also some notations from this paper). We remark that our context necessitates another definition of the "invariant form" and of the integration process which is utilized for the proof of the fact that the characteristic classes are independent of the connection.

I express my profound gratitude to Prof. Enzo Martinelli.

§ 2. GENERAL CONSIDERATIONS

2.1. In the sections § 2–5 the following algebraic structures are fixed.

a) Let the following conditions be simultaneously satisfied:

(1) *a*) R = *commutative ring with* 1,
b) $\mathfrak{F}$ = *commutative* R–*algebra with* 1.

(2) *a*) H, P, T = *Lie* R–*algebras*,
b) $(\mathcal{E}) \equiv 0 \to H \xrightarrow{\iota} P \xrightarrow{\pi} T \to 0$ *exact sequence*,
c) ι, π = *Lie* R–*algebra homomorphisms*.

CONVENTION: we identify H with $\iota H \subset P$.

(3) *a*) H, P, T = $\mathfrak{F}$–*modules* (which induce the R–structures modules 2[a])
b) $\iota\,\pi$ = $\mathfrak{F}$–*homomorphisms*,
c) *there exists a* $\mathfrak{F}$–*homomorphism* $\nabla : T \to P$, *such that* $\pi \cdot \nabla = I_T$,

(4) *a*) $\mathfrak{F}$ = T–*module*; *hence*, $\mathfrak{F}$ *is a* P–*module by* π.
b) $X_1(f_1 \cdot f_2) = (X_1 f_1) \cdot f_2 + f_1(X_1 f_2)$, $\forall f_1, f_2 \in \mathfrak{F}$, $\forall X_1 \in T$
c) $[f_1 X_1, X_2] = f_1 [X_1, X_2] - (\pi(X_2)) f_1 \cdot X_1$ *for* $\forall f_1 \in \mathfrak{F}$, $\forall X_1, X_2 \in P$.

REMARK 1. i) H *is an ideal in* P *and hence a* P–*module by the adjoint representation of* P *over* H:

$$Xh = [X, h] \text{ for } X \in P, h \in H.$$

ii) *For any* $f \in \mathfrak{F}$, $X_1, X_2 \in T$, *we deduce* (*from* (4) *and* π = *epimorphism*)

$$[f X_1, X_2] = f [X_1, X_2] - (X_2 f) X_1 \qquad (4\,c')$$

iii) *For any* $f \in \mathfrak{F}$, $X_1 \in P$, $X_2 \in H$

$$[f X_1, X_2] = f [X_1, X_2].$$

2.2. If M, N are A–modules (A a commutative ring) let be $C_A^n(M, N)$ the A–module of all A–n linear, skew-symmetric functions on M with values in N. If M is, in addition, a Lie algebra over A, let be $d_A^n : C_A^n(M, N) \to \to C_A^{n+1}(M, N)$ defined as follows ($f \in C_A^n(M, N)$, $X_i \in M$):

$$(df)(X_1, \cdots, X_{n+1}) = \sum_{1 \le i \le n+1} (-1)^{i+1} X_i f(X_1, \cdots, \hat{X}_i, \cdots, X_{n+1}) +$$
$$+ \sum_{1 \le i < j \le n+1} (-1)^{i+j} f([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j \cdots X_{n+1}).$$

REMARK 2. *By* (4), (4 c'), *we have*

$$d_R^n C_{\mathfrak{F}}^n(P, H) \subset C_{\mathfrak{F}}^{n+1}(P, H)$$
$$d_R^n C_{\mathfrak{F}}^n(T, \mathfrak{F}) \subset C_{\mathfrak{F}}^{n+1}(T, \mathfrak{F}).$$

Hence, $\{C^n(T, \mathfrak{F}), d_R^n\}_{n \in \mathbf{N}}$ *and* $\{C_{\mathfrak{F}}^n(P, H), d_R^n\}_{n \in \mathbf{N}}$ *are cochain complexes.*

DEFINITION 1. *Let be*:

$$H_{\mathfrak{F}}^*(T, \mathfrak{F}) = H_*\{C_{\mathfrak{F}}^n(T, \mathfrak{F}), d_R^n\}_n. \tag{5}$$

If $f \in C_{\mathfrak{F}}^n(T, M)$, *let be* $\pi^* f \in C_{\mathfrak{F}}^n(P, M)$ *such defined*:

$$(\pi^* f)(X_1, \cdots, X_n) = f(\pi X_1, \cdots, \pi X_n), \; X_i \in P.$$

Any element of $\pi^* C_{\mathfrak{F}}^n(T, M)$ *will be called "a basic form over* P*".*

§ 3. "GEOMETRIC" CONSIDERATIONS

DEFINITION 2. *The splitting* ∇ (see (3)) *will be called a "connection" in the extension* ($\mathcal{E}$).

DEFINITION 3. *The function* $\omega \in C_{\mathfrak{F}}^1(P, H)$ *such defined*

$$\omega(X) = X - \nabla\pi X, \qquad X \in P \tag{6}$$

will be called the "connection form" of the connection $\nabla \cdot$ H *will be called the space of "vertical" elements and* $\nabla T \subset P$ *the space of "horizontal" elements.*

DEFINITION 4. i) *For any* $X_1, X_2 \in P$, *let be*

$$\Omega_\nabla^*(X_1, X_2) = [X_1, \omega(X_2)] - [X_2, \omega(X_1)] - \omega([X_1, X_2]) - - [\omega(X_1), \omega(X_2)]. \tag{7}$$

ii) *For any* $X_1, X_2 \in T$, *let be*

$$\Omega_\nabla(X_1, X_2) = \nabla[X_1, X_2] - [\nabla X_1, \nabla X_2]. \tag{7'}$$

Ω_∇^* *will be called the "spatial curvature" of* ∇ *and* Ω_∇ *the "basic" curvature of* ∇.

PROPOSITION 1.

(8)
i) $\Omega_\nabla \in C^2_{\mathfrak{F}}(T, H)$

ii) $\Omega^*_\nabla = \pi^* \Omega_\nabla$; *hence* $\Omega^*_\nabla \in C^2_{\mathfrak{F}}(P, H)$ *and*

iii) $\Omega^*_\nabla (X_1, X_2) = \nabla\pi [X_1, X_2] - [\nabla\pi X_1, \nabla\pi X_2]$.

Proof. Immediate.

NOTATION. $[\omega, \omega] \in C^2_{\mathfrak{F}}(P, H)$ *and* $[\Omega^*_\nabla, \omega] \in C^3_{\mathfrak{F}}(P, H)$ *are defined as follows*:

$$[\omega, \omega](X_1, X_2) = [\omega(X_1), \omega(X_2)]$$

for any $X_1, X_2 \in P$.

$$[\Omega^*_\nabla, \omega](X_1, X_2, X_3) = [\Omega^*_\nabla(X_1, X_2), \omega(X_3)] - [\Omega^*_\nabla(X_1, X_3), \omega(X_2)] + [\Omega^*_\nabla(X_2, X_3), \omega(X_1)]$$

for any $X_1, X_2, X_3 \in P$.

PROPOSITION 2. (*Structure equation*).

$$d\omega = \Omega^*_\nabla + [\omega, \omega] \tag{9}$$

Proof. Immediate from the definitions.

DEFINITION 5. *If* $\varphi \in C^n_{\mathfrak{F}}(T, E)$, *the "covariant derivative" of* φ, $D_\nabla \varphi \in C^{n+1}_{\mathfrak{F}}(T, E)$, *is*

$$(D_\nabla \varphi)(X_1, \cdots, X_{n+1}) = (d(\pi^* \varphi))(\nabla X_1, \cdots, \nabla X_{n+1}), \quad X_i \in T.$$

PROPOSITION 3. ("*Bianchi identity*")

(10)
i) $d\Omega^*_\nabla = -[\Omega^*_\nabla, \omega]$

ii) $D\Omega_\nabla = 0$.

Proof. i) $(d\Omega^*_\nabla)(X_1, X_2, X_3) = [X_1, \Omega^*_\nabla(X_2, X_3)]_{(p)} - \Omega^*_\nabla([X_1, X_2], X_3)_{(p)} =$
((p) denotes cyclic permutation of indices 1, 2, 3)

$$= -[\Omega^*_\nabla, \omega](X_1, X_2, X_3) + [\nabla\pi X_1, \nabla\pi[X_2, X_3] - [\nabla\pi X_2, \nabla\pi X_3]]_{(p)} -$$
$$- (\nabla\pi [[X_1, X_2], X_3]_{(p)} - [\nabla\pi [X_1, X_2], \nabla\pi X_3]_{(p)}) =$$
$$= -[\Omega^*_\nabla, \omega](X_1, X_2, X_3) - [\nabla\pi X_1, [\nabla\pi X_2, \nabla\pi X_3]]_{(p)} -$$
$$- \nabla\pi [[X_1, X_2], X_3]_{(p)} = -[\Omega^*_\nabla, \omega](X_1, X_2, X_3)$$

(by Jacobi identity).

ii) It follows immediately from i) and Definition 5, because $\omega(\nabla X) = 0$ for $\forall X \in T$.

§ 4. Constant invariant forms on H

DEFINITION 6. *Let be* $\varphi : \underbrace{\mathrm{H} \mathrm{X} \cdots \mathrm{X} \mathrm{H}}_{n \text{ times}} \to \mathcal{F}$ *a* $\mathcal{F}$*–n linear function. We call* φ *" constant invariant" function if and only if, for any* $\mathrm{X} \in \mathrm{P}$ *and* $h_1, \cdots, h_n \in \mathrm{H}$,

$$(\pi(\mathrm{X}))\, \varphi(h_1, \cdots, h_n) = \sum_{\alpha=1}^{n} \varphi(h_1, \cdots, [\mathrm{X}, h_\alpha], \cdots, h_n).$$

Let $\mathrm{I}^n_{\mathcal{F}}(\mathrm{H}, \mathcal{F})$ *be the space of all constant invariant forms on* H.

NOTATIONS. *Let be* $(m) = (m_1, \cdots, m_n) \in \mathbf{N}^n$, $m = \sum_{\alpha=1}^{n} m_\alpha$, $f(\alpha) = \sum_{i=1}^{\alpha-1} m_i$. *Let* P_m *denote the symmetric group on* $\{1, 2, \cdots, m\}$, *and*

$$\mathrm{P}_{(m)} = \{\mathcal{C} \mid \mathcal{C} \in \mathrm{P}_m,\ 1 \leq \mathcal{C}(f(\alpha) + 1) < \cdots < \mathcal{C}(f(\alpha) + k) < \cdots$$
$$\cdots < \mathcal{C}(f(\alpha + 1) - 1) < m \qquad 1 \leq k \leq m_\alpha,\ 1 \leq \alpha \leq n\}.$$

If $\mathrm{X} = (\mathrm{X}_1, \cdots, \mathrm{X}_m) \in \mathrm{M}^m$, *where* M *is a set, we denote, for* $\mathcal{C} \in \mathrm{P}_{(m)}$ $1 \leq \alpha \leq n$,

$$(\mathcal{C}\mathrm{X})_\alpha = (\mathrm{X}_{\mathcal{C}(f(\alpha)+1)}, \cdots, \mathrm{X}_{\mathcal{C}(f(\alpha+1)-1)}) \in \mathrm{M}^{m_\alpha}$$

DEFINITION 7. *If* $\varphi \in \mathrm{I}^n_{\mathcal{F}}(\mathrm{H}, \mathcal{F})$ *and* $\omega_i \in \mathrm{C}^{mi}_{\mathcal{F}}(\mathrm{P}, \mathrm{H})$, *we define* $\varphi(\omega_1, \cdots, \omega_n) \in \mathrm{C}^m_{\mathcal{F}}(\mathrm{T}, \mathcal{F})$ *such*:

$$(11)\qquad \varphi(\omega_1, \cdots, \omega_n)(\mathrm{X}_1, \cdots, \mathrm{X}_m) = \sum_{\mathcal{C} \in \mathrm{P}_{(m)}} \operatorname{sig} \mathcal{C}\ \varphi(\omega_1((\mathcal{C}\mathrm{X})_1), \cdots, \omega_n((\mathcal{C}\mathrm{X})_n)).$$

If $\varphi_1 \in \mathrm{I}^{n_1}_{\mathcal{F}}(\mathrm{H}, \mathcal{F})$, $\varphi_2 \in \mathrm{I}^{n_2}_{\mathcal{F}}(\mathrm{H}, \mathcal{F})$, *then the* $(n_1 + n_2)$*–linear function over* H, *with values in* $\mathcal{F}$, $\varphi_1 \cdot \varphi_2$, *defined by the formula*:

$$(12)\qquad (\varphi_1 \cdot \varphi_2)(h_1, \cdots, h_{n_1+n_2}) =$$
$$= \sum_{1 \leq i_1 < i_2 < \cdots < i_{n_1} \leq n_1+n_2} \varphi_1(h_{i_1}, \cdots, h_{i_{n_1}})\, \varphi_2(h_1, \cdots, \hat{h}_{i_1}, \cdots, \hat{h}_{i_{n_1}}, \cdots, h_{n_1+n_2})$$

is an element of $\mathrm{I}^{n_1+n_2}_{\mathcal{F}}(\mathrm{H}, \mathcal{F})$.

COROLLARY 4. $\mathrm{I}^*_{\mathcal{F}}(\mathrm{H}, \mathcal{F}) = \bigoplus_{n \in \mathbf{N}} \mathrm{I}^n(\mathrm{H}, \mathcal{F})$ *is a* R*–algebra.*

PROPOSITION 5. *Let be* $\varphi \in \mathrm{I}^n_{\mathcal{F}}(\mathrm{H}, \mathcal{F})$ *and* $\omega_i \in \mathrm{C}^{m_i}_{\mathcal{F}}(\mathrm{T}, \mathrm{H})$, $1 \leq i \leq n$. *We define* $\tilde{\varphi}(\omega_1, \cdots, \omega_n) \in \mathrm{C}^m_{\mathcal{F}}(\mathrm{T}, \mathcal{F})$ $(m = \Sigma m_i)$, *by the formula*:

$$(13)\qquad \tilde{\varphi}(\omega_1, \cdots, \omega_n)(\mathrm{X}_1, \cdots, \mathrm{X}_m) = \varphi(\pi^* \omega_1, \cdots, \pi^* \omega_n)(\nabla \mathrm{X}_1, \cdots, \nabla \mathrm{X}_m),\ \mathrm{X}_i \in \mathrm{T}.$$

Then, we have:

$$(14)\qquad d\tilde{\varphi}(\omega_1, \cdots, \omega_n) = \sum_{\alpha=1}^{n} (-1)^{f(\alpha)} \varphi(\pi^* \omega_1, \cdots, \pi^*(\mathrm{D}\omega_\alpha), \cdots, \pi\omega_n).$$

Proof. The proof is direct utilizing the definitions.

REMARK 3. *If* $\omega_i \in C^{mi}_{\mathfrak{F}}(P, H)$, *but* ω_i *are not basic forms, then a relation*:

$$(14') \qquad d\varphi(\omega_1, \cdots, \omega_n) = \Sigma (-1)^{f(\alpha)} \varphi(\omega_1, \cdots, d\omega_\alpha, \cdots, \omega_n)$$

is generally false.

§ 5. CHARACTERISTIC CLASSES OF EXTENSIONS

THEOREM 6. i) *If* $\varphi \in I^n_{\mathfrak{F}}(H, \mathfrak{F})$, *then* (see (13))

$$(15) \qquad \tilde{\varphi}(\Omega_\nabla, \cdots, \Omega_\nabla) \in C^{2n}_{\mathfrak{F}}(T, \mathfrak{F})$$

is a closed form.

ii) *If* $1/k \in R$, $2 \leq k \leq 2n$, *then the homology class of* $\tilde{\varphi}(\Omega_\nabla, \cdots, \Omega_\nabla)$ *in* $\mathcal{H}^{2n}_{\mathfrak{F}}(T, \mathfrak{F})$ *is independent of the connection* ∇.

Proof. i) By the Proposition 5, and Bianchi identity, we have:

$$d\tilde{\varphi}(\Omega_\nabla, \cdots, \Omega_\nabla) = \sum_{\alpha=1}^{n} \varphi(\Omega^*_\nabla, \cdots, \pi^*(D_\nabla \Omega_\nabla), \cdots, \Omega^*_\nabla) = 0.$$

ii) Let ∇_0 and ∇_1 be two connections in $(\mathcal{E})$ and (ω_0, Ω^*_0), (ω_1, Ω^*_1) the corresponding connections and curvature forms.

The fact that $\tilde{\varphi}(\Omega_1, \cdots, \Omega_1) - \tilde{\varphi}(\Omega_0, \cdots, \Omega_0)$ is a coboundary, will be a consequence of the Lemmas 8–11.

We change in (1), (2), (3) resp. $R, \mathfrak{F}, H, P, T$ by resp. $R[t]$, $\mathfrak{F}[t]$, $H[t]$, $P[t]$, $T[t]$, where t is an indeterminate.

If A is a commutative ring with 1, so $A[t]$ does by the operation: $(a_1 t^r)(a_2 t^s) = (a_1 a_2) t^{r+s}$, $a_1, a_2 \in A$. If M is a A–module, then $M[t]$ is an $A[t]$–module by the operation $(at^r)(mt^s) = (am) t^{r+s}$, $a \in A$, $m \in M$. If L is a Lie A–algebra, then $L[t]$ is a Lie $A[t]$–algebra: $[l_1 t^r, l_2 t^s] = [l_1, l_2] t^{r+s}$, $(l_1, l_2 \in L)$. With these generic definitions we introduce the corresponding structures in the new objects and we extend in the natural manner the morphisms ι, π, (we denote them $(\iota[t], \pi[t])$). Now the new algebraic structures satisfy the conditions (1), (2), (3 *a*, *b*), (4). We denote by $\mathcal{E}[t]$ the new extension.

We construct now a connection ∇_t in $\mathcal{E}[t]$. It is sufficient to define it on $T \subset T[t]$; we define:

$$(16) \qquad \nabla_t(X) = \nabla_1(X)\, t + \nabla_0(X)(1 - t), \qquad X \in T$$

Then:

$$\pi[t]\, \nabla_t(X) = \pi\nabla_1(X)\, t + \pi\nabla_0(X)(1 - t) = X,$$

and hence ∇_t is really a connection.

LEMMA 7. i) *The connection form* ω_t *of the connection* ∇_t *is*

(17) $$\omega_t = \omega_1 t + \omega_0 (1 - t) ,$$

ii) *The spatial curvature* Ω_t^* *of* ∇_t *is*:

(18) $$\Omega_t^* = \Omega_0^* + \{d(\omega_1 - \omega_0) + 2[\omega_0 , \omega_0] - [\omega_0 , \omega_1] - [\omega_1 , \omega_0]\} t - \\ - [\omega_1 - \omega_0 , \omega_1 - \omega_0] t^2 .$$

Proof. i) is immediate,
ii) it follows from i^0 and the structure equation.

LEMMA 8. *The coefficients of* t , t^2 *from* (18) *are basic forms.*

Proof. If $X \in H$, then $(\omega_1 - \omega_0)(X) = 0$; hence $\omega_1 - \omega_0$ is a basic form and also $[\omega_1 - \omega_0 , \omega_1 - \omega_0]$. We know that Ω_t^* and Ω_0^* are basic forms; hence, the remainder term is a basic form, too.

We perform now a "differential" and "integral" calculus in this context.

Let A be a commutative ring, M an A–module and t an indeterminate. We define the derivative:

(19) $$\frac{d}{dt} : M[t] \to M[t] \quad , \quad mt^r \mapsto (rm) t^r \qquad \text{for} \quad m \in M .$$

LEMMA 9. *If* M , N *are two* A*–modules, and* $\varphi : \underbrace{M X \cdots X M}_{n \text{ times}} \to N$ *is* A*–multilinear, let* $\varphi_t : M[t] X \cdots X M[t] \to N[t]$ *be the* $A[t]$*–prolongation of* φ. *For any* $Q_1 , \cdots , Q_n \in M[t]$,

(20) $$\frac{d}{dt} \varphi(Q_1 , \cdots , Q_n) = \sum_{\alpha=1}^{n} \varphi(Q_1 , \cdots , \frac{d}{dt} Q_\alpha , \cdots , Q_n)$$

Proof. Obvious.

Let be $V_0 , V_1 : M[t] \to M$ the A–homomorphisms as defined:

(21) $$V_0(mt^r) = \begin{cases} m , & \text{for } r = 0 \\ 0 , & \text{for } r > 0 \end{cases} , \qquad m \in M ,$$
$$V_1(mt^r) = m , \qquad m \in M .$$

Let $n \in \mathbf{N}$ be a fixed number. We suppose $\frac{1}{2} , \frac{1}{3} , \cdots , \frac{1}{n+1} \in A$, and let be the A–module:

$$M[t]_n = \{Q , Q \in M[t] , \quad \text{grade} \quad Q \leq n\} .$$

We define the definite integration $\int_0^1 * \, dt$ as being the A–homomorphism

(22) $$\int_0^1 * \, dt : M[t]_n \to M$$

(22′) $$\int_0^1 mt^r \, dt = \frac{m}{r+1} , \qquad r \leq n , \quad m \in M .$$

LEMMA 10 ("*Newton–Leibnitz*" *formula*). *If* $Q \in M[t]_{n+1}$, *then*

$$\int_0^1 \left(\frac{d}{dt} Q\right) dt = V_1(Q) - V_0(Q). \tag{23}$$

Proof. It follows immediately from (21), (22′).

LEMMA 11. *Let be* $Q \in C^*_{\mathfrak{F}}(T, \mathfrak{F})[t]_{2n}$. *Then*

$$d\int_0^1 Q\, dt = \int_0^1 (dQ)\, dt. \tag{24}$$

Proof. Obvious.

We pass now to the proof of ii) Theorem 6. We have from (17)

$$\frac{d}{dt}\omega_t = \omega_1 - \omega_0, \tag{25}$$

which is a basic form.

We have in consequence:

$$d \sum_{\alpha=1}^{n} \tilde{\varphi}\left(\Omega_t, \cdots, \Omega_t, \frac{d}{dt}\omega_t, \Omega_t, \cdots, \Omega_t\right) = \tag{26}$$

(formula (14))

$$= \sum_{1 \le i < \alpha \le n} \varphi\left(\Omega_t^*, \cdots, \pi^* D_{\underset{(i)}{\nabla_t}} \Omega_t, \cdots, \Omega_t^*, \frac{d}{dt}\omega_t, \Omega_t^*, \cdots, \Omega_t^*\right) +$$

$$+ \sum_{\alpha=1}^{n} \varphi\left(\Omega_t^*, \cdots, \Omega_t^*, \pi^* D_{\nabla_t}\left(\frac{d}{dt}\omega_t\right), \Omega_t^*, \cdots, \Omega_t^*\right) -$$

$$- \sum_{1 \le \alpha < i \le n} \varphi\left(\Omega_t^*, \cdots, \Omega_t^*, \frac{d}{dt}\omega_t, \Omega_t^*, \cdots, \pi^* D_{\nabla_t}\Omega_t, \cdots, \Omega_t^*\right) =$$

(Bianchi identity and (25))

$$= \sum_{\alpha=1}^{n} \varphi(\Omega_t^*, \cdots, \Omega_t^*, \pi^* D_{\nabla_t}(\omega_1 - \omega_0), \Omega_t^*, \cdots, \Omega_t^*).$$

By the other hand, we have, from (8), (9), (16), (18)

$$\pi^* D_{\nabla_t}(\omega_1 - \omega_0) = \frac{d}{dt}\Omega_t^*. \tag{27}$$

Hence, from (26), (27) and (20), we deduce

$$d \sum_{\alpha=1}^{n} \tilde{\varphi}\left(\Omega_t, \cdots, \frac{d}{dt}\omega_t, \cdots, \Omega_t\right) = \sum_{\alpha=1}^{n} \tilde{\varphi}\left(\Omega_t, \cdots, \Omega_t, \frac{d}{dt}\Omega_t, \cdots, \Omega_t\right) = \tag{28}$$

$$= \frac{d}{dt}\tilde{\varphi}(\Omega_t, \cdots, \Omega_t).$$

We have, from (28), (23), (24)

$$V_1(\tilde{\varphi}(\Omega_t,\cdots,\Omega_t)) - V_0(\tilde{\varphi}(\Omega_t,\cdots,\Omega_t)) = \tag{29}$$

$$= \int_0^1 d\sum_{\alpha=1}^{n}\tilde{\varphi}\left(\Omega_t,\cdots,\Omega_t,\frac{d}{dt}\omega_t,\Omega_t,\cdots,\Omega_t\right)dt =$$

$$= d\int_0^1 \sum_{\alpha=1}^{n}\tilde{\varphi}\left(\Omega_t,\cdots,\Omega_t,\frac{d}{dt}\omega_t,\Omega_t,\cdots,\Omega_t\right)dt\,.$$

We observe that the integrand is a polynomial of degree $\leq 2n-2$, and from (18), (21), we have:

$$V_1(\tilde{\varphi}(\Omega_t,\cdots,\Omega_t)) = \tilde{\varphi}(\Omega_1,\cdots,\Omega_1)$$
$$V_0(\tilde{\varphi}(\Omega_t,\cdots,\Omega_t)) = \tilde{\varphi}(\Omega_0,\cdots,\Omega_0)\,,$$

hence, from (29)

$$\tilde{\varphi}(\Omega_1,\cdots,\Omega_1) - \tilde{\varphi}(\Omega_0,\cdots,\Omega_0) = d\int_0^1 \sum_{\alpha=1}^{n}\tilde{\varphi}\left(\Omega_t,\cdots,\frac{d}{dt}\omega_t,\cdots,\Omega_t\right)dt\,,$$

and the Theorem 6 is completely established.

DEFINITION 8. Let be $\varphi \in I^n_{\mathfrak{F}}(H,\mathfrak{F})$ and Ω the curvature form of a connection for $(\mathcal{E})$. If $\frac{1}{n}\in R$, $n \in \mathbf{N}$, then we denote by $[\varphi(\Omega)]$ the homology class (in $\mathcal{H}^{2n}_{\mathfrak{F}}(T,\mathfrak{F})$) of $\tilde{\varphi}(\Omega,\cdots,\Omega)\in C^{2n}_{\mathfrak{F}}(T,\mathfrak{F})$. The set $\{[\varphi(\Omega)]\,|\,\varphi\in I^*_{\mathfrak{F}}(H,\mathfrak{F})\}\subset \mathcal{H}^*_{\mathfrak{F}}(T,\mathfrak{F})$ will be called the characteristic ring of the extension $(\mathcal{E})$ (see Corollary 4).

COROLLARY 12. *If there exists a connection* ∇ *for the extension* $(\mathcal{E})$ *which is a Lie algebra homomorphism* $\left(\textit{in the presence of the condition } \frac{1}{n}\in R,\ n\in\mathbf{N}\right)$ *then the characteristic ring of* $(\mathcal{E})$ *is trivial.*

See the following Nota II.