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Francois Sigrist, Ulrich Suter

On symplectic Stiefel manifolds

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Geometria. — *On symplectic Stiefel manifolds.* Nota di FRANÇOIS SIGRIST e ULRICH SUTER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota preventiva vengono studiate certe rappresentazioni fra varietà simplettiche di Stiefel.

1. INTRODUCTION

Let $\mathrm{S}p(n)$ be the symplectic group and $\mathrm{W}_{n,k} = \mathrm{S}p(n)/\mathrm{S}p(n-k)$ the symplectic Stiefel manifold. For $k > l$, one has an obvious map $p : \mathrm{W}_{n,k} \to \mathrm{W}_{n,l}$, which is a fiber map with fiber $\mathrm{W}_{n-l,k-l}$. The purpose of this paper is to give a complete description of the values of n, k and l for which the map p has a cross-section. Details will appear elsewhere.

The corresponding problem for the orthogonal Stiefel manifolds is already completely solved (Adams, 1962; Eckmann–Whitehead, 1963), as is the unitary case (Adams–Walker, 1965; Suter, 1966), the most famous contribution being Adams' solution of the vector field problem on spheres [1]. For the symplectic Stiefel manifolds, the explicit results previously known can be stated as follows:

(i) The map $p : \mathrm{W}_{n,2} \to \mathrm{W}_{n,1} = \mathrm{S}^{4n-1}$ has a cross-section if and only if n is a multiple of 24. This result is due to I. M. James [7].

(ii) For $k > l \geq 2$, the map $p : \mathrm{W}_{n,k} \to \mathrm{W}_{n,l}$ does not have a cross-section. For this result see [9, page 203].

We shall dispose of the remaining cases, i.e. the cases with $l = 1$, $k > 2$, by the following Theorem.

2. MAIN THEOREM

For any positive integer q and any prime p let $\nu_p(q)$ be the exponent of p in the prime power decomposition of q. We then have:

THEOREM. *The symplectic Stiefel fibring* $p : \mathrm{W}_{n,k} \to \mathrm{W}_{n,1}$ *has a cross-section if and only if one of the following two equivalent conditions holds*:

(I) *For each integer* j *with* $0 \leq j \leq k-1$ *the coefficient* a_j *of* z^j *in*

$$\left[\frac{2}{\sqrt{z}} \operatorname{ARSh} \frac{\sqrt{z}}{2}\right]^{2n} = \sum_{j=0}^{\infty} a_j z^j = 1 - \frac{n}{12} z + \cdots$$

is an *integer* *if* j *is even and an* *even integer* *if* j *is odd.*

(*) Nella seduta dell'8 aprile 1972.

(II) *n is a multiple of the integer c_k, called quaternionic James number, which is defined by its decomposition into prime powers as follows*:

$$\nu_2(c_k) = \max_s (2k - 1, 2s + \nu_2(s)), \quad 1 \leq s \leq k - 1$$

$$\nu_p(c_k) = \max_t (t + \nu_p(t)), \qquad 1 \leq t \leq \left[\frac{2k-1}{p-1}\right], \; p \text{ odd} < 2k$$

$$\nu_p(c_k) = 0, \qquad p \text{ odd} > 2k.$$

We observe in particular that c_k is divisible by all primes less than $2k$. The number c_2 is equal to 24, this is James' result mentioned above.

In the course of the proof of our Theorem we show that c_k is either equal to b_{2k} or to $1/2\, b_{2k}$, where b_{2k} is the (known) complex James number. A closer study of the relation between c_k and b_{2k} shows: For k odd, c_k is always equal to $1/2\, b_{2k}$. On the other hand if k is even, there are cases with $c_k = b_{2k}$ and such with $c_k = 1/2\, b_{2k}$; the distribution turns out to be rather irregular, however for approximatively 73 % of all cases we have $c_{2m} = b_{4m}$.

3. Outline of the Proof

The Theorem is proved using techniques developed by Adams and Walker in the unitary case [3].

Let KO (X) resp. KU (X) be real resp. complex K–Theory of a finite CW–complex X, and let $J : KO(X) \to J(X)$ be Atiyah's J–homomorphism [4]. By HP^{k-1} we denote the $(k-1)$–dimensional quaternionic right projective space. The following two Theorems are then the starting point for our study.

THEOREM [James, 6]. *There exists a positive integer c_k, such that the Stiefel fibring $p : W_{n,k} \to W_{n,1}$ has a cross-section if and only if n is a multiple of c_k.* (The integer c_k is the so called quaternionic James number).

THEOREM [Atiyah, 4]. *Let $\alpha \in KU(HP^{k-1})$ be the canonical 2–dimensional complex Hopf-bundel and let $r\alpha \in KO(HP^{k-1})$ be its realification. The quaternionic James number c_k is the order of the element $J(r\alpha)$ in $J(HP^{k-1})$.*

To determine the order of $J(r\alpha)$ we use the groups $J''(HP^{k-1})$ resp. $J'(HP^{k-1})$ as defined in [3], which constitute a computable upper resp. lower bound of the group $J(HP^{k-1})$. We prove:

LEMMA 1. *For the space HP^m one has the isomorphisms*

$$J''(HP^m) \overset{\theta''}{\cong} J(HP^m) \overset{\theta'}{\cong} J'(HP^m).$$

Proof. The first isomorphism follows from the now proved Adams conjecture [Quillen, 8]. Working with the cofibration $HP^{m-1} \to HP^m \to S^{4m}$ one gets $\theta' \circ \theta'' : J''(HP^m) \cong J'(HP^m)$ by induction on m, in the same way as in [3, Lemma 6.1].

The above Lemma enables us to work with $J''(HP^{k-1})$ or $J'(HP^{k-1})$ instead of $J(HP^{k-1})$, whatever is more appropriate in a given context. We first deal with $J'(HP^{k-1})$.

By definition [3] we have $J'(X) = KO(X)/V(X)$, where $V(X)$ is the subgroup of elements $\delta \in KO(X)$, such that $sh(\delta) = ch \circ c(1 + \gamma)$ for some element $\gamma \in \widetilde{KO}(X)$. (Here $sh : KO(X) \to 1 + \sum_{s>0}^{\infty} H^{4s}(X; Q)$ is the characteristic class corresponding to the power series of $\dfrac{e^{y/2} - e^{-y/2}}{y} = \dfrac{2\,\mathrm{Sh}(y/2)}{y}$, $c : KO(X) \to KU(X)$ is complexification and $ch : KU(X) \to H^*(X; Q)$ is the chern character). The order c_k of the element $J'(r\alpha)$ in $J'(HP^{k-1})$ is therefore the smallest positive integer n for which

$$sh(-n \cdot r\alpha) = [sh(r\alpha)]^{-n} \in ch \circ c(KO(HP^{k-1})).$$

(We regard $-n$ for convenience). The canonical map

$$g : CP^{2k-1} \to HP^{k-1},$$

where CP^{2k-1} is complex projective space, induces an *injection* in ordinary cohomology, and we obtain, computing $KO(HP^{k-1}) \xrightarrow{c} KU(HP^{k-1}) \xrightarrow{g!} KU(CP^{2k-1})$, the following two results:

(i) The image of the map

$$g^* \circ ch \circ c : KO(HP^{k-1}) \to H^*(CP^{2k-1}; Q) = Q[y] \pmod{y^{2k}}$$

is the free abelian group generated by $1, 2[2\,\mathrm{Sh}(y/2)]^2, [2\,\mathrm{Sh}(y/2)]^4, \cdots, \cdots, \varepsilon_j[2\,\mathrm{Sh}(y/2)]^{2j}, \cdots, \varepsilon_{k-1}[2\,\mathrm{Sh}(y/2)]^{2(k-1)} \pmod{y^{2k}}$, where $\varepsilon_j = 1$ if j is even and $\varepsilon_j = 2$ if j is odd.

(ii) $g^* \circ sh(-n \cdot r\alpha) = [sh \circ g!(r\alpha)]^{-n} = \left[\dfrac{2\,\mathrm{Sh}(y/2)}{y}\right]^{-2n} \pmod{y^{2k}}$.

In $Q[y] \pmod{y^{2k}}$ one has a unique relation

$$(R_n) : \left[\frac{y}{2\,\mathrm{Sh}(y/2)}\right]^{2n} = \sum_{j=0}^{k-1} q_j \cdot \varepsilon_j [2\,\mathrm{Sh}(y/2)]^{2j} \pmod{y^{2k}},$$

where $q_0 = 1, q_1, \cdots, q_{k-1}$ are rational numbers depending on n, and consequently we deduce:

LEMMA 2. *The order c_k of the element $J'(r\alpha)$ in $J'(HP^{k-1})$ is the smallest integer n, for which the coefficients $q_0 = 1, q_1, \cdots, q_{k-1}$ in (R_n) are integers.*

By an obvious transformation of power series we derive from Lemma 2 part I of our Theorem. To get part II we work with $J''(HP^{k-1})$ and prove:

LEMMA 3. *The quaternionic James number c_k is either equal to the complex James number b_{2k} or to $1/2\, b_{2k}$.*

Proof. We first show, that the element $J''_C(\alpha) \in J''_C(HP^{k-1})$ has the same order as $J''(r\beta) \in J''(CP^{2k-2})$; here $r\beta$ is the realification of the canonical complex linebundle β over the complex projective space CP^{2k-2}. One has $KU(HP^{k-1}) = \mathbf{Z}[\sigma] \pmod{\sigma^k}$, where $\sigma = \alpha - 2$. On the other hand one knows, that $KO(CP^{2k-2}) = \mathbf{Z}[\omega] \pmod{\omega^k}$, where $\omega = r\beta - 2$ (see [3]). The homomorphism $g_1^* : KU(HP^{k-1}) \to KU(CP^{2k-2})$ induced by $CP^{2k-2} \subset CP^{2k-1} \xrightarrow{g} HP^{k-1}$ is given by $g_1^*(\alpha) = \beta + \bar\beta$, and the complexification $c : KO(CP^{2k-2}) \to KU(CP^{2k-2})$ is determined by $c(r\beta) = \beta + \bar\beta$. Since both g_1^* and c are injective and compatible with the ψ–operations one obtains, slightly abusing notation, a ψ–ring isomorphism $c^{-1} \circ g_1^* = A : KU(HP^{k-1}) \cong \cong KO(CP^{2k-2})$ (1). With the induced isomorphism $J''(A) : J''_C(HP^{k-1}) \cong \cong J''(CP^{2k-2})$ (see [2]) we deduce: The order of $J''_C(\alpha)$ is equal to the order of $J''(r\beta)$ which is b_{2k-1} [3]; by [5, p. 344] one has $b_{2k-1} = b_{2k}$. The homomorphism $J''(r) : J''_C(HP^{k-1}) \to J''(HP^{k-1})$ (see [3, Appendix]) maps $J''_C(\alpha)$ onto $J''(r\alpha)$ and it follows, that the order of $J''(r\alpha)$, i.e. the integer c_k, divides b_{2k}. But b_{2k} is a factor of $2c_k$ [6, 1.5] and Lemma 3 is proved.

From Lemma 3 we obtain $\nu_p(c_k) = \nu_p(b_{2k})$ for odd primes p; $\nu_p(b_{2k})$ is explicitly determined in [5, p. 343]. To complete the proof of part II of our Theorem it remains therefore to determine $\nu_2(c_k)$. Referring to part I we proceed as follows:

The function $f(z) = \left[\frac{2}{\sqrt{z}} \operatorname{ARSh} \frac{\sqrt{z}}{2}\right]^{2n}$ satisfies the functional equation

$$f(z) = \left(1 + \frac{z}{4}\right)^n \cdot f(4z + z^2).$$

This implies the following relation for the coefficients $a_0 = 1, a_1, a_2, \cdots, a_j, \cdots$ of the power series of f:

$$\binom{n}{j} + \sum_{i=1}^{j-1} \binom{n+i}{j-i} 4^{2i} \cdot a_i = 4^j (1 - 4^j) \cdot a_j, \; j = 1, 2, \cdots. \tag{1}$$

In other words: The coefficients $a_1, a_2, \cdots$ are the solution of the above system of linear equations, and c_k is the smallest positive integer n, for which the first $k-1$ elements $a_1, \cdots, a_{k-1}$ of this solution are alternatively even and integral.

Working with (1) and proving some simple number-theoretical Lemmas one shows now by induction on k, that $\nu_2(c_k)$ is as stated in our main Theorem.

(1) This elegant way of deriving the ψ–ring isomorphism $KU(HP^{k-1}) \cong KO(CP^{2k-2})$ was communicated to us by J.F. Adams.

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