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Cesàro—Hilbert-Schmidt operators

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Analisi funzionale. — *Cesàro–Hilbert–Schmidt operators.* Nota di GH. CONSTANTIN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si generalizza la classe degli operatori di Hilbert–Schmidt introducendo la classe degli operatori di Cesàro–Hilbert–Schmidt.

1. One of the important classes of operators studied intensively in functional analysis is the class of Hilbert–Schmidt operators. Our aim in the present Note is to give a generalization of this class of operators introducing the Cesàro–Hilbert–Schmidt operators. We consider only the case of Hilbert spaces.

2. Let H_1 and H_2 be separable Hilbert spaces.

DEFINITION 2.1. An operator $T : H_1 \to H_2$ is called of Cesàro–Hilbert–Schmidt type if for every orthonormal basis of H_1, $\{e_i\}_{i=1}^{\infty}$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} \sum_{k=1}^{n} \| Te_k \| \right)^2 < \infty . \tag{1}$$

LEMMA 2.1. *Every Hilbert–Schmidt operator is of Cesàro–Hilbert–Schmidt type.*

Proof. We have, from well-known properties of such operators, that

$$\| T \|_2^2 = \sum_{k=1}^{\infty} \| Te_k \|^2 .$$

Since for $p \geq 1$ we have

$$\left(\sum_{k=1}^{n} | a_k | \right)^p \leq n^{p-1} \sum_{k=1}^{n} | a_k |^p$$

then

$$\left(\frac{1}{n} \sum_{k=1}^{n} \| Te_k \| \right)^2 \leq \frac{1}{n} \sum_{k=1}^{n} \| Te_k \|^2$$

and thus

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} \sum_{k=1}^{n} \| Te_k \| \right)^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{k=1}^{n} \| Te_k \|^2 \right) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \| T \|_2^2 < \infty$$

which proves our assertion.

(*) Nella seduta dell'8 aprile 1972.

We prove now the following

LEMMA 2.2. *Every Cesàro-Hilbert-Schmidt operator is compact.*

Proof. We use the Pelczyński device [3]. For every orthonormal basis we have that $\|Te_k\| \to 0$ for $k \to \infty$.

If this is not so, then there would exist $\varepsilon > 0$ and an orthonormal sequence $\{e_n\}_{n=1}^{\infty}$ such that $\|Te_n\| > \varepsilon$ for $n = 1, 2, \cdots$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} \sum_{k=1}^{n} \|Te_k\| \right)^2 > \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} n\varepsilon \right)^2$$

which is a clear contradiction.

With this Lemma we can prove the following:

THEOREM 2.1. *The necessary and sufficient condition such that* $T : H_1 \to H_2$ *be of Cesàro-Hilbert-Schmidt type is that* T *be of the form*

$$Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle h_n \tag{2}$$

where $\{e_n\}_{n=1}^{\infty}$, $\{h_n\}_{n=1}^{\infty}$ *are orthonormal basis of* H_1, H_2 *respectively,* $\lambda_n \geq 0$ *and such that*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} \sum_{k=1}^{n} \lambda_k \right)^2 < \infty . \tag{3}$$

Proof. If T is of Cesàro-Hilbert-Schmidt type then from Lemma 2.2, T is compact and therefore T has the form

$$Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle h_n$$

where $\{e_n\}_{n=1}^{\infty}$ are eigenvectors corresponding to the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of the positive compact part S of the polar decomposition $T = US$. Since $Te_k = \lambda_k h_k$ it is clear that the condition is necessary.

Conversely, if this is so, then T is compact. Indeed, $\lambda_n \to 0$, $n \to \infty$ since in the contrary case there exists $\varepsilon > 0$ such that $\lambda_k > \varepsilon$, $k = 1, 2, \cdots$ and thus $\sum_{k=1}^{n} \lambda_k > \varepsilon n$ and therefore

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} \sum_{k=1}^{n} \lambda_k \right)^2 > \varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

which is a contradiction.

From the relation (2) and since $\lim_{n \to \infty} \lambda_n = 0$ we conclude that T is a compact operator for which the relation (1) is satisfied.

PROPOSITION 2.1. *The set of Cesàro-Hilbert-Schmidt operators is a linear space.*

Proof. If S and T are Cesàro–Hilbert–Schmidt operators then $S+T$ is also of the same type. Indeed, if we denote by $\sigma_n(\|Se_k\|) = \frac{1}{n}\sum_{k=1}^{n}\|Se_k\|$ then

$$\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}\|(S+T)\,e_k\|\right)^2\right]^{1/2} \leq$$

$$\leq \left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}(\|Se_k\|+\|Te_k\|)\right)^2\right]^{1/2} =$$

$$= \left[\sum_{n=1}^{\infty}\left(\frac{\sigma_n(\|Se_k\|)}{n^{1/2}}+\frac{\sigma_n(\|Te_k\|)}{n^{1/2}}\right)^2\right]^{1/2} \leq$$

$$\leq \left[\sum_{n=1}^{\infty}\left(\frac{\sigma_n(\|Se_k\|)}{n^{1/2}}\right)^2\right]^{1/2} + \left[\sum_{n=1}^{\infty}\left(\frac{\sigma_n(\|Te_k\|)}{n^{1/2}}\right)^2\right]^{1/2} =$$

$$= \left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}\|Se_k\|\right)^2\right]^{1/2} + \left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}\|Te_k\|\right)^2\right]^{1/2} < \infty .$$

It is easy to see that λT is a Cesàro–Hilbert–Schmidt operator for every scalar λ and T a Cesàro–Hilbert–Schmidt operator.

PROPOSITION 2.2. *If* S *is a bounded linear operator and* T *a Cesàro–Hilbert–Schmidt operator then* ST *is a Cesàro–Hilbert–Schmidt operator.*

Proof. If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of H_1 we have

$$\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}\|STe_k\|\right)^2\right]^{1/2} \leq \|S\| \left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}\|Te_k\|\right)^2\right]^{1/2} < \infty .$$

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