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On a unilateral problem for the Navier-Stokes equations. Nota II

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Analisi matematica. — *On a unilateral problem for the Navier–Stokes equations* (*). Nota II di GIOVANNI PROUSE, presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si dà la dimostrazione dei due Teoremi enunciati nella Nota I.

2. Let us give the proof of Theorem 1 stated in the preceding Note I.

Denote by β a penalization operator relative to the convex set K, i.e. an operator such that

$$K = \{\vec{z} \in V_1 \mid \beta(\vec{z}) = 0\}. \tag{2.1}$$

As is well known (see, for example, Lions [1], ch. 3, § 5) we can set

$$\beta(\vec{z}) = J(\vec{z} - P_K \vec{z}), \tag{2.2}$$

where J is a duality operator from V_1 to V_1' (dual of V_1) and P_K is the projection operator from V_1 to K. We shall assume to have chosen a duality operator relative to the function $\Phi(r) = r$, which, by definition, satisfies therefore the conditions

$$\langle J(\vec{v}), \vec{v}\rangle = \| J(\vec{v})\|_{V_1'} \|\vec{v}\|_{V_1} \tag{2.3}$$

$$\| J(\vec{v})\|_{V_1'} = \|\vec{v}\|_{V_1} \qquad \forall \vec{v} \in V_1, \tag{2.4}$$

where the symbol $\langle\ ,\ \rangle$ denotes the duality between V_1' and V_1.

Since V_1 and V_1' (which are Hilbert spaces) are reflexive and have uniformly convex norms, such a duality operator is uniquely defined and is hemicontinuous and strictly monotone.

Let $\{\vec{g}_j\}$ be a basis in $V_1 \cap H^2(\Omega)$ and set

$$\vec{u}_n(t) = \sum_{j=1}^{n} \alpha_{jn}(t)\, \vec{g}_j. \tag{2.5}$$

Consider the system of n ordinary differential equations

$$(\vec{u}_n'(t), \vec{g}_j)_{V_0} + \mu(\vec{u}_n(t), \vec{g}_j)_{V_1} + b(\vec{u}_n(t), \vec{u}_n(t), \vec{g}_j) - \langle \vec{f}(t), \vec{g}_j\rangle + \tag{2.6}$$

$$+ n\langle \beta(\vec{u}_n(t)), \vec{g}_j\rangle + \int_{\Gamma_1} \left(\varphi(x,t) - \frac{1}{2}|\vec{u}_n(x,t)|^2\right)\vec{g}_j(x) \times \vec{\nu}\, d\Gamma_1 = 0 \quad (j = 1, \cdots, n)$$

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(**) Nella seduta dell'11 marzo 1972.

with the initial conditions

$$\vec{u}_n(0) = 0 . \tag{2.7}$$

It is easily seen that (2.6), (2.7) admit a solution for $t > 0$ sufficiently small; we shall now prove that a solution $\vec{u}_n(t)$ of (2.6), (2.7) exists on the whole of $[0, T]$.

Multiplying (2.6) by $\alpha_{jn}(t)$, adding and integrating between 0 and $t \in [0, T]$ we obtain, bearing in mind (2.7),

$$\frac{1}{2} \|\vec{u}_n(t)\|_{V_0}^2 + \int_0^t \Big\{ \mu \|\vec{u}_n(\eta)\|_{V_1}^2 + b(\vec{u}_n(\eta), \vec{u}_n(\eta), \vec{u}_n(\eta)) - \langle \vec{f}(\eta), \vec{u}_n(\eta)\rangle + \\ + n \langle \beta(\vec{u}_n(\eta)), \vec{u}_n(\eta)\rangle + \int_{\Gamma_1} \Big(\varphi(x, \eta) - \frac{1}{2} |\vec{u}_n(x, \eta)|^2 \Big) \vec{u}_n(x, \eta) \times \vec{\nu} \, d\Gamma_1 \Big\} d\eta = 0 \tag{2.8}$$

and consequently, by (1.16),

$$\frac{1}{2} \|\vec{u}_n(t)\|_{V_0}^2 + \int_0^t \Big\{ \mu \|\vec{u}_n(\eta)\|_{V_1}^2 - \langle \vec{f}(\eta), \vec{u}_n(\eta)\rangle + n \langle \beta(\vec{u}_n(\eta)), \vec{u}_n(\eta)\rangle + \\ + \int_{\Gamma_1} \varphi(x, \eta) \vec{u}_n(x, \eta) \times \vec{\nu} \, d\Gamma_1 \Big\} d\eta \leq 0 . \tag{2.9}$$

Since β is monotone with $\beta(0) = 0$, we have

$$\langle \beta(\vec{u}_n), \vec{u}_n \rangle \geq 0 ; \tag{2.10}$$

hence, by (2.9),

$$\frac{1}{2} \|\vec{u}_n(t)\|_{V_0}^2 + \int_0^t \Big\{ \mu \|\vec{u}_n(\eta)\|_{V_1}^2 - \langle \vec{f}(\eta), \vec{u}_n(\eta)\rangle + \\ + \int_{\Gamma_1} \varphi(x, \eta) \vec{u}_n(x, \eta) \times \vec{\nu} \, d\Gamma_1 \Big\} d\eta \leq 0 \tag{2.11}$$

and also, denoting by γ the operator trace on Γ,

$$\frac{1}{2} \|\vec{u}_n(t)\|_{V_0}^2 + \mu \int_0^t \|\vec{u}_n(\eta)\|_{V_1}^2 \, d\eta \leq \\ \leq \int_0^t \Big\{ \|\vec{f}(\eta)\|_{V_1'} \|\vec{u}_n(\eta)\|_{V_1} + \|\varphi(\eta)\|_{L^2(\Gamma_1)} \|\gamma \vec{u}_n(\eta)\|_{L^2(\Gamma_1)} \Big\} d\eta \leq \\ \leq \int_0^t \Big\{ \|\vec{f}(\eta)\|_{V_1'} \|\vec{u}_n(\eta)\|_{V_1} + c \|\varphi(\eta)\|_{L^2(\Gamma_1)} \|\vec{u}_n(\eta)\|_{V_1} \Big\} d\eta . \tag{2.12}$$

From (2.12) it follows that

$$\|\vec{u}_n(t)\|_{V_0} \leq M_1 \quad , \quad \int_0^T \|\vec{u}_n(t)\|_{V_1}^2 \, dt \leq M_2 , \tag{2.13}$$

M_1 and M_2 being independent of n. Hence the solution $\vec{u}_n(t)$ exists on the whole interval $[0, T]$ and

$$\lim_{n\to\infty} \vec{u}_n(t) \underset{L^2(0,T;V_1)}{=} \vec{u}(t) \quad , \quad \lim_{n\to\infty} \vec{u}_n(t) \underset{L^\infty(0,T;V_0)}{=} \vec{u}(t) \tag{2.14}$$ (1)

in the weak and weak-star topologies respectively. On the other hand, by (2.9), (2.13),

$$\int_0^T \langle \beta(\vec{u}_n(t)), \vec{u}_n(t)\rangle \, dt \leq$$

$$\leq \frac{1}{n} \int_0^T \left\{ |\langle \vec{f}(t), \vec{u}_n(t)\rangle| + \left| \int_{\Gamma_1} \varphi(x,t)\, \vec{u}_n(x,t) \times \vec{\nu} \, d\Gamma_1 \right| \right\} dt \leq \frac{M_3}{n}$$

and, consequently,

$$\lim_{n\to\infty} \int_0^T \langle \beta(\vec{u}_n(t)), \vec{u}_n(t)\rangle \, dt = 0 . \tag{2.15}$$

We now recall (see Lions [1] ch. 3 th. 5.1) that

$$\langle J(\vec{v} - P_K \vec{v}), P_K \vec{v}\rangle \geq 0 \qquad \forall \vec{v} \in V_1 ;$$

hence, by (2.3), (2.4),

$$\int_0^T \langle \beta(\vec{u}_n(t)), \vec{u}_n(t)\rangle \, dt = \int_0^T \langle J(\vec{u}_n(t) - P_K \vec{u}_n(t)), \vec{u}_n(t)\rangle \, dt = \tag{2.16}$$

$$= \int_0^T \langle J(\vec{u}_n(t) - P_K \vec{u}_n(t)), \vec{u}_n(t) - P_K \vec{u}_n(t)\rangle \, dt +$$

$$+ \int_0^T \langle J(\vec{u}_n(t) - P_K \vec{u}_n(t)), P_K \vec{u}_n(t)\rangle \, dt \geq \int_0^T \|\vec{u}_n(t) - P_K \vec{u}_n(t)\|_{V_1}^2 \, dt .$$

From (2.15), (2.16) it follows that

$$\lim_{n\to\infty} \int_0^T \|\vec{u}_n(t) - P_K \vec{u}_n(t)\|_{V_1}^2 \, dt = 0$$

(1) We shall always, for simplicity, again denote by $\{\vec{u}_n\}$ subsequences selected from $\{\vec{u}_n\}$.

which implies

$$\lim_{n\to\infty} \|\beta(\vec{u}_n)\|_{L^2(0,T;V_1')} = \lim_{n\to\infty} \|J(\vec{u}_n - P_K \vec{u}_n)\|_{L^2(0,T;V_1')} = \tag{2.17}$$

$$= \lim_{n\to\infty} \|\vec{u}_n - P_K \vec{u}_n\|_{L^2(0,T;V_1)} = 0 .$$

Denoting by $\vec{\varphi}$ an element of $L^2(0, T; V_1)$, we have, by (2.14), (2.15), (2.17) and the monotonicity of β,

$$\int_0^T - \langle \beta(\vec{\varphi}(t)), \vec{u}(t) - \vec{\varphi}(t)\rangle \, dt = \tag{2.18}$$

$$= \lim_{n\to\infty} \int_0^T \langle \beta(\vec{u}_n(t)) - \beta(\vec{\varphi}(t)), \vec{u}_n(t) - \vec{\varphi}(t)\rangle \, dt \geq 0 .$$

Hence, setting $\vec{\varphi}(t) = \vec{u}(t) + \lambda\vec{\psi}(t)$, $\vec{\psi}(t) \in L^2(0, T; V_1)$, $\lambda > 0$, it follows from (2.18) that

$$\int_0^T \langle \beta(\vec{u}(t) - \lambda\vec{\psi}(t)), \vec{\psi}(t)\rangle \, dt \leq 0 \qquad \forall \vec{\psi}(t) \in L^2(0, T; V_1)$$

and, letting $\lambda \to 0$ and bearing in mind that β is hemicontinuous,

$$\int_0^T \langle \beta(\vec{u}(t)), \vec{\psi}(t)\rangle \, dt \leq 0 \qquad \forall \vec{\psi}(t) \in L^2(0, T; V_1) . \tag{2.19}$$

Therefore

$$\beta(\vec{u}) = 0 , \tag{2.20}$$

i.e. $\vec{u}(t) \in K$ a.e. on $[0, T]$.

Let us now differentiate (2.6); we obtain

$$(\vec{u}_n''(t), \vec{g}_j)_{V_0} + \mu(\vec{u}_n'(t), \vec{g}_j)_{V_1} + b(\vec{u}_n'(t), \vec{u}_n(t), \vec{g}_j) + \tag{2.21}$$

$$+ b(\vec{u}_n(t), \vec{u}_n'(t), \vec{g}_j) - \langle \vec{f}'(t), \vec{g}_j\rangle + n\langle(\beta(\vec{u}_n(t)))', \vec{g}_j\rangle +$$

$$+ \int_{\Gamma_1} \left\{ \left(\frac{\partial\varphi(x,t)}{\partial t} - \sum_{j=1}^{2} u_{jn}(x,t) \frac{\partial u_{jn}(x,t)}{\partial t} \right) \vec{g}_j(x) \times \vec{\nu} \, d\Gamma_1 \right\} dt = 0$$

$$(j = 1, \cdots, n).$$

Multiplying (2.21) by $\alpha'_{jn}(t)$, adding and integrating between 0 and $t \in [0, T]$, we have

$$(2.22) \quad \frac{1}{2}\|\vec{u}'_n(t)\|^2_{V_0} - \frac{1}{2}\|\vec{u}'_n(0)\|^2_{V_0} + \int_0^t \Big\{ \mu \|\vec{u}'_n(\eta)\|^2_{V_1} + b(\vec{u}'_n(\eta), \vec{u}_n(\eta), \vec{u}'_n(\eta)) +$$

$$+ b(\vec{u}_n(\eta), \vec{u}'_n(\eta), \vec{u}'_n(\eta)) + n \langle (\beta(\vec{u}_n(\eta)))', \vec{u}'_n(\eta)\rangle - \langle \vec{f}'(\eta), \vec{u}'_n(\eta)\rangle +$$

$$+ \int_{\Gamma_1} \left(\frac{\partial \varphi(x,\eta)}{\partial \eta} - \sum_{j=1}^{2} u_{jn}(x,\eta) \frac{\partial u_{jn}(x,\eta)}{\partial \eta} \right) \frac{\partial \vec{u}_n(x,\eta)}{\partial \eta} \times \vec{\nu}\, d\Gamma_1 \Big\} d\eta = 0 .$$

Observe that, by the monotonicity of β,

$$\langle \beta(\vec{\varphi}(t+h)) - \beta(\vec{\varphi}(t)), \vec{\varphi}(t+h) - \vec{\varphi}(t)\rangle \geq 0 \qquad \forall \vec{\varphi}(t) \in H^1(0, T; V_1)$$

and consequently, since β satisfies a Lipschitz condition (V_1 being a Hilbert space),

$$(2.23) \qquad \langle (\beta(\vec{\varphi}(t)))', \vec{\varphi}'(t)\rangle \geq 0 .$$

Moreover, by well known embedding theorems,

$$|b(\vec{u}'_n, \vec{u}_n, \vec{u}'_n)| \leq c_1 \|\vec{u}'_n\|^2_{L^4(\Omega)} \|\vec{u}_n\|_{V_1} \leq c_2 \|\vec{u}'_n\|_{V_0} \|\vec{u}'_n\|_{V_1} \|\vec{u}_n\|_{V_1} \leq$$

$$\leq \frac{\mu}{8} \|\vec{u}'_n\|^2_{V_1} + c_3 \|\vec{u}'_n\|^2_{V_0} \|\vec{u}_n\|_{V_1} ,$$

$$|b(\vec{u}_n, \vec{u}'_n, \vec{u}'_n)| \leq c_4 \|\vec{u}_n\|_{L^4(\Omega)} \|\vec{u}'_n\|_{V_1} \|\vec{u}'_n\|_{L^4(\Omega)} \leq c_5 \|\vec{u}_n\|_{L^4(\Omega)} \|\vec{u}'_n\|^{3/2}_{V_1} \|\vec{u}'_n\|^{1/2}_{V_0} \leq$$

$$\leq \frac{\mu}{8} \|\vec{u}'_n\|^2_{V_1} + c_6 \|\vec{u}'_n\|^2_{V_0} \|\vec{u}_n\|^4_{L^4(\Omega)} ,$$

(2.24)

$$\int_{\Gamma} |\vec{u}_n|\, |\vec{u}'_n|^2 d\Gamma \leq \|\gamma \vec{u}_n\|_{L^3(\Gamma)} \|\gamma \vec{u}'_n\|^2_{L^3(\Gamma)} \leq c_7 \|\vec{u}_n\|_{V_{2/3}} \|\vec{u}'_n\|^2_{V_{2/3}} \leq$$

$$\leq c_8 \|\vec{u}_n\|^{1/3}_{V_0} \|\vec{u}_n\|^{2/3}_{V_1} \|\vec{u}'_n\|^{2/3}_{V_0} \|\vec{u}'_n\|^{4/3}_{V_1} \leq \frac{\mu}{8} \|\vec{u}'_n\|^2_{V_1} + c_9 \|\vec{u}_n\|_{V_0} \|\vec{u}_n\|^2_{V_1} \|\vec{u}'_n\|^2_{V_0} ,$$

$$\int_{\Gamma_1} |\varphi'|\, |\vec{u}'_n|\, d\Gamma_1 \leq \frac{1}{2} \|\varphi'\|^2_{L^2(\Gamma_1)} + \frac{1}{2} \|\gamma \vec{u}'_n\|_{L^2(\Gamma_1)} \leq$$

$$\leq \frac{1}{2} \|\varphi'\|^2_{L^2(\Gamma_1)} + c_{10} \|\vec{u}'_n\|^2_{V_{2/3}} \leq \frac{1}{2} \|\varphi'\|^2_{L^2(\Gamma_1)} + \frac{\mu}{8} \|\vec{u}'_n\|^2_{V_1} + c_{11} \|\vec{u}'_n\|^2_{V_0} .$$

From (2.22), (2.23), (2.24) it follows that

$$\frac{1}{2}\|\vec{u}_n'(t)\|_{V_0}^2 + \frac{\mu}{2}\int_0^t \|\vec{u}_n'(\eta)\|_{V_1}^2\, d\eta \leq \tag{2.25}$$

$$\leq \frac{1}{2}\|\vec{u}_n'(0)\|_{V_0}^2 + \frac{1}{2}\int_0^t \Big\{\|\vec{f}'(\eta)\|_{V_0}^2 + \|\varphi'(\eta)\|_{L^2(\Gamma_1)}^2 +$$

$$+ c_{12}\|\vec{u}_n'(\eta)\|_{V_0}^2\left[1 + \|\vec{u}_n(\eta)\|_{V_1}^2 + \|\vec{u}_n(\eta)\|_{L^4(\Omega)}^4 + \|\vec{u}_n(\eta)\|_{V_0}\|\vec{u}_n(\eta)\|_{V_1}^2\right]\Big\}\, d\eta$$

and also

$$\|\vec{u}_n'(t)\|_{V_0}^2 \leq \left[\|\vec{u}_n'(0)\|_{V_0}^2 + \int_0^T \left(\|\vec{f}'(t)\|_{V_0}^2 + \|\varphi'(t)\|_{L^2(\Gamma_1)}^2\right) dt\right] \cdot \tag{2.26}$$

$$\cdot \exp c_{12}\int_0^t \Big\{1 + \|\vec{u}_n(\eta)\|_{V_1}^2 + \|\vec{u}_n(\eta)\|_{L^4(\Omega)}^4 + \|\vec{u}_n(\eta)\|_{V_0}\|\vec{u}_n(\eta)\|_{V_1}^2\Big\}\, d\eta\,.$$

Hence, by (2.13) (since $L^4(0, T; L^4(\Omega)) \supset L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$),

$$\|\vec{u}_n'(t)\|_{V_0}^2 \leq \|\vec{u}_n'(0)\|_{V_0}^2 + M_3\,. \tag{2.27}$$

Setting, on the other hand, in (2.6) $t = 0$, we obtain, by the assumptions made,

$$(\vec{u}_n'(0), \vec{g}_j)_{V_0} = \langle \vec{f}(0), \vec{g}_j\rangle = (\vec{f}(0), \vec{g}_j)_{V_0} \qquad (j = 1, \cdots, n), \tag{2.28}$$

that is

$$\|\vec{u}_n'(0)\|_{V_0} \leq \|\vec{f}(0)\|_{V_0}\,. \tag{2.29}$$

It follows from (2.27), (2.29) that

$$\|\vec{u}_n'(t)\|_{V_0} \leq M_4 \tag{2.30}$$

and, by (2.25),

$$\int_0^T \|\vec{u}_n'(t)\|_{V_1}^2\, dt \leq M_5\,, \tag{2.31}$$

M_4 and M_5 being independent of n.

Hence

$$\lim_{n\to\infty} \vec{u}_n'(t) \underset{L^2(0,T;V_1)}{=} \vec{u}'(t) \quad , \quad \lim_{n\to\infty} \vec{u}_n'(t) \underset{L^\infty(0,T;V_0)}{=} \vec{u}'(t) \tag{2.32}$$

in the weak and weak-star topologies respectively.

By (2.14), (2.20), (2.32) the limit function $\vec{u}(t)$ satisfies condition I_A). It remains to be proved that $\vec{u}(t)$ satisfies also II_A).

Let

$$(2.33) \qquad \vec{h}(t) = \sum_{j=1}^{\infty} \lambda_j(t)\,\vec{g}_j$$

be any function $\in H^1(0,T;V_1 \cap H^2(\Omega))$, with $\vec{h}(x,t) > 0$ for $x \in \Gamma_2$, $t \in [0,T]$; setting

$$(2.34) \qquad \vec{h}_p(t) = \sum_{j=1}^{p} \lambda_j(t)\,\vec{g}_j\,,$$

it is obvious that, when p is $\geq \bar{p}$ sufficiently large, $\vec{h}_p(t) \in H^1(0,T;V_1 \cap H^2(\Omega))$ and $\in K \ \ \forall t \in [0,T]$. Assuming then that $p \geq \bar{p}$ and setting $\gamma_j(t) = \lambda_j(t)$ when $j \leq p$, $\gamma_j(t) = 0$ when $j > p$, let us multiply (2.6) by $\alpha_{jn}(t) - \gamma_j(t)$; if $n \geq p$ we obtain, adding and integrating over $[0,T]$,

$$(2.35) \qquad \int_0^T \Big\{ (\vec{u}_n'(t)\,,\vec{u}_n(t) - \vec{h}_p(t))_{V_0} + \mu(\vec{u}_n(t)\,,\vec{u}_n(t) - \vec{h}_p(t))_{V_1} +$$
$$+\, b(\vec{u}_n(t)\,,\vec{u}_n(t)\,,\vec{u}_n(t) - \vec{h}_p(t)) - (\vec{f}(t)\,,\vec{u}_n(t) - \vec{h}_p(t))_{V_0} +$$
$$+\, n\,\langle \beta(\vec{u}_n(t))\,,\vec{u}_n(t) - \vec{h}_p(t)\rangle +$$
$$+ \int_{\Gamma_1} \Big(\varphi(x,t) - \frac{1}{2}\,|\vec{u}_n(x,t)|^2\Big)(\vec{u}_n(x,t) - \vec{h}_p(x,t)) \times \vec{\nu}\, d\Gamma_1 \Big\}\, dt = 0\,.$$

Since $\vec{h}_p(t) \in K$,

$$\langle \beta(\vec{u}_n)\,,\vec{u}_n - \vec{h}_p\rangle = \langle \beta(\vec{u}_n) - \beta(\vec{h}_p)\,,\vec{u}_n - \vec{h}_p\rangle \geq 0$$

and consequently

$$(2.36) \qquad \int_0^T \Big\{ (\vec{u}_n'(t)\,,\vec{u}_n(t) - \vec{h}_p(t))_{V_0} + \mu(\vec{u}_n(t)\,,\vec{u}_n(t) - \vec{h}_p(t))_{V_1} +$$
$$+\, b(\vec{u}_n(t)\,,\vec{u}_n(t)\,,\vec{u}_n(t) - \vec{h}_p(t)) - (\vec{f}(t)\,,\vec{u}_n(t) - \vec{h}_p(t))_{V_0} +$$
$$+ \int_{\Gamma_1} \Big(\varphi(x,t) - \frac{1}{2}\,|\vec{u}_n(x,t)|^2\Big)(\vec{u}_n(x,t) - \vec{h}_p(x,t)) \times \vec{\nu}\, d\Gamma_1 \Big\}\, dt \leq 0\,.$$

Observe now that by (2.13), (2.31),

$$\|\vec{u}_n(t)\|_{H^1(0,T;V_1)} \leq M_6 ;$$

hence, by well known trace and compactness Theorems,

$$\lim_{n\to\infty} \gamma\vec{u}_n(t) \underset{L^3(0,T;L^3(\Gamma))}{=} \gamma\vec{u}(t) \quad , \quad \lim_{n\to\infty} \vec{u}_n(t) \underset{L^4(0,T;L^4(\Omega))}{=} \vec{u}(t) \tag{2.37}$$

in the strong topologies. Letting $n \to \infty$ in (2.36), we obtain then, by (2.14), (2.32), (2.37),

$$\begin{aligned}\int_0^T \Big\{ & (\vec{u}'(t), \vec{u}(t) - \vec{h}_p(t))_{V_0} + \mu(\vec{u}(t), \vec{u}(t) - \vec{h}_p(t))_{V_1} + \\ & + b(\vec{u}(t), \vec{u}(t), \vec{u}(t) - \vec{h}_p(t)) - (\vec{f}(t), \vec{u}(t) - \vec{h}_p(t))_{V_0} + \\ & + \int_{\Gamma_1} \Big(\varphi(x,t) - \frac{1}{2} |\vec{u}(x,t)|^2\Big)(\vec{u}(x,t) - \vec{h}_p(x,t)) \times \vec{\nu}\, d\Gamma_1 \Big\} dt \leq 0\end{aligned} \tag{2.38}$$

$\forall \vec{h}_p(t)$ given by (2.34), with $p \geq \bar{p}$.

Since the set of such functions is dense in $L^2(0, T; K)$, the relation

$$\begin{aligned}\int_0^T \Big\{ & (\vec{u}'(t), \vec{u}(t) - \vec{h}(t))_{V_0} + \mu(\vec{u}(t), \vec{u}(t) - \vec{h}(t))_{V_1} + \\ & + b(\vec{u}(t), \vec{u}(t), \vec{u}(t) - \vec{h}(t)) - (\vec{f}(t), \vec{u}(t) - \vec{h}(t))_{V_0} + \\ & + \int_{\Gamma_1} \Big(\varphi(x,t) - \frac{1}{2} |\vec{u}(x,t)|^2\Big)(\vec{u}(x,t) - \vec{h}(x,t)) \times \vec{\nu}\, d\Gamma_1 \Big\} dt \leq 0\end{aligned} \tag{2.39}$$

will hold $\forall \vec{h}(t) \in L^2(0, T; K)$.

The existence of a weak solution is therefore proved.

Let us now show that the solution $\vec{u}(t)$ is unique. Assume, in fact, that $\vec{v}(t)$ is a second solution, i.e. is such that $\vec{v}(t) \in L^2(0, T; V_1)$, $v'(t) \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$, $\vec{v}(t) \in K$ $\forall t \in [0, T]$ and satisfies the inequality

$$\begin{aligned}\int_0^T \Big\{ & (\vec{v}'(t), \vec{v}(t) - \vec{h}(t))_{V_0} + \mu(\vec{v}(t), \vec{v}(t) - \vec{h}(t))_{V_1} + \\ & + b(\vec{v}(t), \vec{v}(t), \vec{v}(t) - \vec{h}(t)) - (\vec{f}(t), \vec{v}(t) - \vec{h}(t))_{V_0} + \\ & + \int_{\Gamma_1} \Big(\varphi(x,t) - \frac{1}{2} |\vec{v}(x,t)|^2\Big)(\vec{v}(x,t) - \vec{h}(x,t)) \times \vec{\nu}\, d\Gamma_1 \Big\} dt \leq 0\end{aligned} \tag{2.40}$$

$\forall \vec{h}(t) \in L^2(0, T; K)$.

Setting in (1.12) and (2.40) respectively $\vec{h}(t) = \vec{v}(t)$ and $\vec{h}(t) = \vec{u}(t)$, adding and denoting by $\vec{w}(t)$ the difference $\vec{u}(t) - \vec{v}(t)$, we obtain

$$(2.41) \qquad \int_0^T \Big\{ (\vec{w}'(t), \vec{w}(t))_{V_0} + \mu \, \|\vec{w}(t)\|^2_{V_1} + b(\vec{u}(t), \vec{u}(t), \vec{w}(t)) -$$

$$- b(\vec{v}(t), \vec{v}(t), \vec{w}(t)) - \frac{1}{2} \int_{\Gamma_1} (|\vec{u}(x,t)|^2 - |\vec{v}(x,t)|^2) \, \vec{w}(x,t) \times \vec{\nu} \, d\Gamma_1 \Big\} \, dt \leq 0$$

and from (2.41) it follows directly, since $\vec{w}(0) = 0$, that $\vec{w}(t) = 0$ in $[0, T]$.

3. We now prove Theorem 2 stated in § 1.

Repeating without any modifications the initial part of the proof given in § 2, we can show that system (2.6) admits a solution $\vec{u}_n(t)$ in $[0, T]$ satisfying the initial condition

$$\vec{u}_n(0) = \vec{v}_n ,$$

where $\vec{v} = \sum_{j=1}^{\infty} \chi_j \vec{g}_j$, $\vec{v}_n = \sum_{j=1}^{n} \chi_j \vec{g}_j$. Moreover,

$$(3.1) \qquad \lim_{n\to\infty} \vec{u}_n(t) \underset{L^2(0,T;V_1)}{=} \vec{u}(t) \quad , \quad \lim_{n\to\infty} \vec{u}_n(t) \underset{L^\infty(0,T;V_0)}{=} \vec{u}(t)$$

in the weak and weak-star topologies respectively and $\vec{u}(t) \in K$ a.e. on $[0, T]$. The limit function $\vec{u}(t)$ therefore satisfies condition I_B).

Let

$$(3.2) \qquad \vec{h}(t) = \sum_{j=1}^{\infty} \lambda_j(t) \, \vec{g}_j$$

be a function $\in H^1(0,T; V_1 \cap H^2(\Omega))$, with $\vec{h}(x,t) > 0$ for $x \in \Gamma_2$, $t \in [0,T]$, $\lambda_j(0) = \chi_j$. Setting

$$(3.3) \qquad \vec{h}_p(t) = \sum_{j=1}^{p} \lambda_j(t) \, \vec{g}_j ,$$

it is obvious that, when $p \geq \bar{p}$ sufficiently large, $\vec{h}_p(t) \in H^1(0, T; V_1 \cap H^2(\Omega))$ and $\in K$ $\forall t \in [0, T]$.

Assuming that $p \geq \bar{p}$ and setting $\gamma_j(t) = \lambda_j(t)$ when $j \leq p$, $\gamma_j(t) = 0$ when $j > p$, we multiply (2.6) (written for $n \geq p$) by $\alpha_{jn}(t) - \gamma_j(t)$, add and integrate over [o, T]; we thus obtain

$$(3.4)\qquad \int_0^T \Big\{ (\vec{u}_n'(t), \vec{u}_n(t) - \vec{h}_p(t))_{V_0} + \mu\, (\vec{u}_n(t), \vec{u}_n(t) - \vec{h}_p(t))_{V_1} + $$
$$+ b(\vec{u}_n(t), \vec{u}_n(t), \vec{u}_n(t) - \vec{h}_p(t)) - \langle \vec{f}(t), \vec{u}_n(t) - \vec{h}_p(t)\rangle +$$
$$+ n\, \langle \beta(\vec{u}_n(t)), \vec{u}_n(t) - \vec{h}_p(t)\rangle +$$
$$+ \int_{\Gamma_1} \Big(\varphi(x, t) - \frac{1}{2}\, |\vec{u}_n(x, t)|^2\Big) (\vec{u}_n(x, t) - \vec{h}_p(x, t)) \times \vec{\nu}\, d\Gamma_1 \Big\}\, dt = 0 .$$

On the other hand, bearing in mind (1.16) and that $\vec{u}_n(0) - \vec{h}_p(0) = = \sum_{j=1}^{n} \chi_j \vec{g}_j - \sum_{j=1}^{p} \chi_j \vec{g}_j = \sum_{j=p+1}^{n} \chi_j \vec{g}_j$ and $\vec{h}_p(t) \in K$, we have

$$b(\vec{u}_n, \vec{u}_n, \vec{u}_n) = \frac{1}{2} \int_{\Gamma_1 \cup \Gamma_2} |\vec{u}_n|^2\, \vec{u}_n \times \vec{\nu}\, d\Gamma \geq \frac{1}{2} \int_{\Gamma_1} |\vec{u}_n|^2\, \vec{u}_n \times \vec{\nu}\, d\Gamma_1 ,$$

$$\int_0^T (\vec{u}_n'(t), \vec{u}_n(t) - \vec{h}_p(t))_{V_0}\, dt = \frac{1}{2}\, \|\vec{u}_n(T) - \vec{h}_p(T)\|_{V_0}^2 -$$
$$(3.5)\qquad - \frac{1}{2}\, \|\vec{u}_n(0) - \vec{h}_p(0)\|_{V_0}^2 + \int_0^T (\vec{h}_p'(t), \vec{u}_n(t) - \vec{h}_p(t))_{V_0}\, dt \geq$$
$$\geq \int_0^T (\vec{h}_p'(t), \vec{u}_n(t) - \vec{h}_p(t))_{V_0}\, dt - \frac{1}{2} \Big\| \sum_{j=p+1}^{n} \chi_j \vec{g}_j \Big\|_{V_0}^2 ,$$

$$\langle \beta(\vec{u}_n), \vec{u}_n - \vec{h}_p\rangle = \langle \beta(\vec{u}_n) - \beta(\vec{h}_p), \vec{u}_n - \vec{h}_p\rangle \geq 0 .$$

Consequently, substituting (3.5) into (3.4),

$$(3.6)\qquad \int_0^T \Big\{ (\vec{h}_p'(t), \vec{u}_n(t) - \vec{h}_p(t))_{V_0} + \mu\, (\vec{u}_n(t), \vec{u}_n(t) - \vec{h}_p(t))_{V_1} -$$
$$- b(\vec{u}_n(t), \vec{u}_n(t), \vec{h}_p(t)) - \langle \vec{f}(t), \vec{u}_n(t) - \vec{h}_p(t)\rangle +$$
$$+ \int_{\Gamma_1} \varphi(x, t)\, (\vec{u}_n(x, t) - \vec{h}_p(x, t)) \times \vec{\nu}\, d\Gamma_1 +$$
$$+ \frac{1}{2} \int_{\Gamma_1} |\vec{u}_n(x, t)|^2\, \vec{h}_p(x, t) \times \vec{\nu}\, d\Gamma_1 \Big\}\, dt \leq \Big\| \sum_{j=p+1}^{\infty} \chi_j \vec{g}_j \Big\|_{V_0}^2 .$$

Observe that, by (3.1), it is obviously

$$\lim_{n\to\infty}\int_0^T \left\{(\vec{h}_p'(t), \vec{u}_n(t)-\vec{h}_p(t))_{V_0} + \mu(\vec{u}_n(t), \vec{u}_n(t)-\vec{h}_p(t))_{V_1} - \langle \vec{f}(t), \vec{u}_n(t)-\vec{h}_p(t)\rangle\right\} dt \geq \int_0^T \left\{(\vec{h}_p'(t), \vec{u}(t)-\vec{h}_p(t))_{V_0} + \mu(\vec{u}(t), \vec{u}(t)-\vec{h}_p(t))_{V_1} - \langle \vec{f}(t), \vec{u}(t)-\vec{h}_p(t)\rangle\right\} dt. \tag{3.7}$$

Moreover, by a Lemma of Hopf, from (3.1) follows that

$$\lim_{n\to\infty} \vec{u}_n(t) \underset{L^2(0,T;V_0)}{=} \vec{u}(t) \tag{3.8}$$

in the strong topology. Hence, observing that $\vec{h}_p(t) \in H^1(0, T; H^2(\Omega)) \subset \subset L^\infty(0, T; L^\infty(\Omega))$,

$$\lim_{n\to\infty} u_{kn} h_{jp} \underset{L^2(0,T;V_0)}{=} u_k h_{jp} \quad , \quad \lim_{n\to\infty} \frac{\partial u_{jn}}{\partial x_k} \underset{L^2(0,T;V_0)}{=} \frac{\partial u_j}{\partial x_k}$$

respectively in the strong and weak topologies and consequently

$$\lim_{n\to\infty}\int_0^T b(\vec{u}_n(t), \vec{u}_n(t), \vec{h}_p(t))\, dt = \lim_{n\to\infty}\int_0^T\int_\Omega \sum_{j,k=1}^3 u_{kn}\frac{\partial u_{jn}}{\partial x_k} h_{jp}\, d\Omega\, dt = \int_0^T\int_\Omega \sum_{j,k=1}^3 u_k \frac{\partial u_j}{\partial x_k} h_{jp}\, d\Omega\, dt = \int_0^T b(\vec{u}(t), \vec{u}(t), \vec{h}_p(t))\, dt. \tag{3.9}$$

Analogously, since

$$\|\gamma\vec{u}_n\|_{L^2(0,T;L^2(\Gamma))} \leq c\|\vec{u}_n\|_{L^2(0,T;V_1)} \leq M, \tag{3.10}$$

we have

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_1} |\vec{u}_n(x,t)|^2\, \vec{h}_p(x,t)\times\vec{\nu}\, d\Gamma_1\, dt = \int_0^T\int_{\Gamma_1} |\vec{u}(x,t)|^2\, \vec{h}_p(x,t)\times\vec{\nu}\, d\Gamma_1\, dt. \tag{3.11}$$

From (3.6), (3.7), (3.9), (3.11) it follows that $\vec{u}(t)$ satisfies the relation

$$(3.12)\quad \int_0^T \Big\{ (\vec{h}_p'(t), \vec{u}(t) - \vec{h}_p(t))_{V_0} + \mu (\vec{u}(t), \vec{u}(t) - \vec{h}_p(t))_{V_1} - b(\vec{u}(t), \vec{u}(t), \vec{h}_p(t)) - \langle \vec{f}(t), \vec{u}(t) - \vec{h}_p(t) \rangle + \int_{\Gamma_1} \varphi(x, t)(\vec{u}(x, t) - \vec{h}_p(x, t)) \times \vec{\nu}\, d\Gamma_1 + \frac{1}{2} \int_{\Gamma_1} |\vec{u}(x, t)|^2 \vec{h}_p(x, t) \times \vec{\nu}\, d\Gamma_1 \Big\} dt \leq \Big\| \sum_{j=p+1}^{\infty} \chi_j \vec{g}_j \Big\|_{V_0}^2 .$$

$\forall \vec{h}_p(t)$ defined by (3.2), (3.3). Letting $p \to \infty$, $\vec{u}(t)$ is therefore a solution of (1.18) $\forall \vec{h}(t)$ given by (3.2). The set of such functions is however dense in the space of test functions considered in condition II_B) and the theorem is therefore proved.

Bibliography

[1] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non lineaires*, Dunod, Paris 1969.