Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A Picone Identity for Fourth Order Differential Equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **52** (1972), n.4, p. 455–456.

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<http://www.bdim.eu/item?id=RLINA_1972_8_52_4_455_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1972.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta dell'8 aprile 1972

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *A Picone Identity for Fourth Order Differential Equations.* Nota di KURT KREITH, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — L'identità di Picone è uno strumento fondamentale nello studio del comportamento oscillatorio delle soluzioni delle equazioni differenziali Sturm–Liouville. Essa è strettamente collegata alla condizione di Legendre nel calcolo delle variazioni. Nella Nota presente si dimostra che un'analoga identità per le equazioni lineari differenziali ordinarie del quart'ordine è collegata pur essa al calcolo delle variazioni.

Picone's identity is a fundamental tool in studying the oscillation properties of second order differential equations. It states that if $y(x)$ and $v(x)$ are solutions of the selfadjoint Sturm–Liouville equations

$$-(r(x)\,y')' + p(x)\,y = 0\,, \tag{1}$$

$$-(\mathrm{R}(x)\,v')' + \mathrm{P}(x)\,v = 0\,, \tag{2}$$

respectively, and if $y(x)/v(x)$ is of class C′, then

$$\frac{d}{dx}\left[\frac{y}{v}(vry' - y\mathrm{R}v')\right] = y'^2(r - \mathrm{R}) + y^2(p - \mathrm{P}) + \mathrm{R}(y' - y/v\,\mathrm{V}')^2. \tag{3}$$

Numerous results regarding the location of zeros of solutions of (1) and (2) follow readily from (3) (see [1], [2]).

Since the oscillation properties of fourth order differential equations have been an area of considerable activity in recent years, an appropriate generalization of (3) to solutions of

$$(r(x)\,y'')'' - (q(x)\,y')' + p(x)\,y = 0 \tag{4}$$

$$(\mathrm{R}(x)\,v'')'' - (\mathrm{Q}(x)\,v')' + \mathrm{P}(x)\,v = 0 \tag{5}$$

(*) Nella seduta del 15 gennaio 1972.

is of interest. While certain extensions of (3) for fourth order equations have been presented by the author [3] and Dunninger [4], neither of these extensions lead to the natural generalization of the classical results which follow from (3).

The identity (8) to be presented below is closely related to an identity established by Leighton [5] in connection with a problem in the calculus of variations. Let $u(x)$ and $v(x)$ be linearly independent solutions of (5) with double zeros at $x = a$ and define

$$\sigma = uv' - u'v$$
$$\tau = u'v'' - u''v'.$$

Then as shown in [5],

$$\frac{d}{dx}\left[\mathrm{R}\sigma'/\sigma y'^2 - 2\,\mathrm{R}\,\tau/\sigma\, yy' + (\mathrm{R}\tau)'/\sigma\, y^2\right] = \mathrm{R}y''^2 + \mathrm{Q}y'^2 + \mathrm{P}y^2 - \mathrm{R}\,(y'' - \sigma'/\sigma y' + \tau/\sigma y)^2 \tag{6}$$

is an identity in y, y', y''. Now, if in addition $y(x)$ is a solution of (4), then

$$\frac{d}{dx}\left[-(ry'')'y + ry''y' + qy'y\right] = ry''^2 + qy'^2 + py^2. \tag{7}$$

Subtracting (6) from (7) yields what appears to be the natural generalization of (3):

$$\frac{d}{dx}\left[-(ry'')'y + ry''y' + qy'y - \mathrm{R}\sigma'/\sigma y'^2 + 2\,\mathrm{R}\tau/\sigma yy' - (\mathrm{R}\tau)'/\sigma y^2\right] = y''^2(r - \mathrm{R}) + y'^2(q - \mathrm{Q}) + y^2(p - \mathrm{P}) + \mathrm{R}\,(y'' - \sigma'/\sigma y' + \tau/\sigma\, y)^2. \tag{8}$$

Using the fact that the conjugate points of $x = a$ with respect to (5) are identical with the zeros of $\sigma(x)$, the following generalized Sturm Theorem follows from (8) in direct analogy with the standard proof in the second order case [2].

THEOREM. *If $y(x)$ is a nontrivial solution of* (4) *with double zeros at $x = a$ and $x = b$ $(a < b)$ and if $r(x) \geq \mathrm{R}(x)$, $q(x) \geq \mathrm{Q}(x)$, $p(x) \geq \mathrm{P}(x)$ for $a \leq x \leq b$, then the first conjugate point of $x = a$ with respect to* (5) *lies in the interval $(a, b]$.*

The above Theorem is also proven in [5], [6] and elsewhere by means of other techniques.

REFERENCES

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[5] W. LEIGHTON, *Quadratic functionals of second order*, « Trans. Amer. Math. Soc. », *151*, 309–322 (1970).
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