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A mathematical model for physical theories Nota II

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Fisica matematica. — *A mathematical model for physical theories* (*).
Nota II di Enzo Tonti, presentata (**) dal Socio B. Finzi.

Riassunto. — In questa Nota si continua l'esame delle proprietà di un modello matematico di una teoria fisica, presentato in una Nota precedente. Tali proprietà riguardano in particolare la formulazione variazionale, l'invertibilità del legame costitutivo, la decomposizione dell'equazione fondamentale in una parte spaziale ed una temporale, nonché la costruzione dello schema duale.

1.1. Introduction

This is the second part of a paper which deals with a mathematical model for physical theories [3]. In this paper we prove a number of mathematical properties that follow from the assumptions given in [3]. In this paper we take away the limitation concerning the linearity of definition and constitutive operators used in the properties shown in the preceding paper.

1.2. Invertible constitutive mappings

Many mathematical properties of the model are based on the possibility to invert the constitutive mapping C. The necessary and sufficient condition is that C be one-to-one. This leads to investigate sufficient conditions in order that C be one-to-one. When C is linear a sufficient condition is that it be positive definite i.e. $\langle \mathrm{C}u , u \rangle > 0$ for $u \neq \vartheta$ (ϑ is the null element of the U–space). This property is frequently met in physical theories.

When C is nonlinear we have the

Theorem 10 (Invertibility Theorem): *a sufficient condition in order that a mapping* C *be one-to-one (and then be invertible) is that* C *be strictly monotone, i.e.*

$$\langle \mathrm{C}(u') - \mathrm{C}(u'') , u' - u'' \rangle > 0 \qquad \text{for} \quad u' \neq u'' \text{ (1)}. \tag{1.2.1}$$

Proof: if C is strictly monotone and $u' \neq u''$ must be $\mathrm{C}(u') \neq \mathrm{C}(u'')$. This assures that two different elements u' and u'' cannot correspond to the same element v and then the mapping is one-to-one. Because the condition of being strictly monotone reduces to that of being positive definite in the linear

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(1) When $>$ is replaced by $\geq$ we have the definition of *monotone* operator.

case, we shall consider in the sequel only strictly monotone operators. What can be said about the inverse of a strictly monotone operator? We have

THEOREM 11: *The inverse of a strictly monotone operator is also a strictly monotone operator.*

Proof: with the position $u = \mathrm{C}^{-1}(v)$ relation (1.2.1) becomes

$$\langle v' - v'', \mathrm{C}^{-1}(v') - \mathrm{C}^{-1}(v'')\rangle > 0 \qquad \text{for } v' \neq v''. \tag{1.2.2}$$

1.3. SPACE AND TIME PART OF THE FUNDAMENTAL MAPPING (2)

When configuration variables depend on space and time coordinates it can happen that the definition operator D be the sum of two operators, generally nonlinear, formed with space and time derivatives respectively. In this case we can decompose the operator D and the set of first kind variables according to the scheme

$$\left[\frac{u_t}{u_s}\right] = \left[\frac{\mathrm{D}_t}{\mathrm{D}_s}\right] \varphi\,. \tag{1.3.1}$$

This amounts to considering the U–space as the sum of two subspaces U_t and U_s i.e. $\mathrm{U} = \mathrm{U}_t \oplus \mathrm{U}_s$. When this happens the balance equation can be written in the form ($\widetilde{\mathrm{D}}$ is the adjoint of D)

$$[\widetilde{\mathrm{D}}_t \mid \widetilde{\mathrm{D}}_s] \left[\frac{v_t}{v_s}\right] = \sigma \qquad \widetilde{\mathrm{D}}_t \to \widetilde{\mathrm{D}}'_{t\varphi} \text{ and } \widetilde{\mathrm{D}}_s \to \widetilde{\mathrm{D}}'_{s\varphi} \text{ in the non linear case} \tag{1.3.2}$$

and the V–space can be conceived as the sum of two subspaces $\mathrm{V} = \mathrm{V}_t \oplus \mathrm{V}_s$.

Moreover the constitutive operator C can often be decomposed according to the scheme

$$\left[\frac{v_t}{v_s}\right] = \left[\begin{array}{c|c} \mathrm{C}_t & 0 \\ \hline 0 & \mathrm{C}_s \end{array}\right] \left[\frac{u_t}{u_s}\right] \tag{1.2.3.}$$

where C_t and C_s can be nonlinear operators.

Under these hypotheses on the decomposition of D and C the fundamental mapping becomes

$$\widetilde{\mathrm{D}}_t \mathrm{C}_t \mathrm{D}_t \varphi + \widetilde{\mathrm{D}}_s \mathrm{C}_s \mathrm{D}_s \varphi = \sigma\,. \tag{1.3.4}$$

(2) In order to have an example to support the mind, the reader can think of the elastodynamic field whose fundamental equation is

$$\left[+\frac{\partial}{\partial t}\right] \rho\, a_{hk} \left[\frac{\partial}{\partial t}\right] u^k + \left[-\nabla^k\right] \mathrm{C}_{hkrs} \left[\frac{1}{2}(\nabla^r u^s + \nabla^s u^r)\right] = f_h$$

(Navier equation) where the operator C_s is Hooke tensor C_{hkrs}, C_t is $\rho\, a_{hk}$, φ is the displacement vector u^k, σ is the body force f_h, a_{hk} the metric tensor, D_s is the symmetrical part of gradient of the displacement vector u^k, D_t the time derivative.

The subspaces U_t, U_s (and V_t, V_s) can be disjointed and conceived as two distinct spaces.

The corresponding scheme is shown in fig. 1

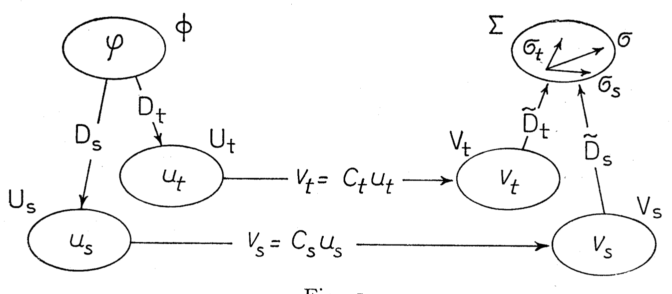

Fig. 1.

The decomposition into a time and space part of the operator C has several mathematical advantages. For example in many physical theories the operator C is not monotone, while C_s is.

Another property is expressed by the following

THEOREM 12: *if* D_s *is a linear operator with dense domain in the* Φ*–space then if* C_s *is monotone the operator* $F_s = \tilde{D}_s C_s D_s$ *is also monotone.*

Proof:

$$(1.3.5) \quad \langle C_s(u') - C_s(u''), u' - u'' \rangle = \langle C_s(D_s\varphi') - C_s(D_s\varphi''), D_s\varphi' - D_s\varphi'' \rangle = \\ = \langle \tilde{D}_s C_s(D_s\varphi') - \tilde{D}_s C_s(D_s\varphi''), \varphi' - \varphi'' \rangle \geq 0 .$$

From this property it follows that the fundamental mapping F written in the form

$$(1.3.6) \qquad F_t \varphi = - F_s \varphi + \sigma$$

has the typical structure of monotonic evolution equations to which many Theorems about existence, uniqueness and continuous dependence on initial data can be applied [1] [4].

1.4. THE POTENTIALS

One of the assumptions of the mathematical model (n. 10) is that the constitutive mapping $C : U \mapsto V$ be symmetric (if linear) or have a symmetric Gateaux derivative (if nonlinear). Such operators enjoy the property that the circulation of the vector $v = C(u)$ along a line in the U–space connecting two fixed points does not depend on the line chosen [5]. In other words the mapping $v = C(u)$ can be regarded as describing a conservative vector field in the U–space. This fact leads us to consider a potential that is a functional defined by

$$(1.4.1) \qquad E[u] = E[u_0] + \int_{\lambda=0}^{\lambda=1} \langle C(\eta), \delta\eta \rangle \quad \text{and then} \quad \delta E[u] = \langle C(u), \delta u \rangle$$

being $\eta = \eta(\lambda)$ so that $\eta(0) = u_0$, $\eta(1) = u$. For this reason the operator C is said to be a *potential operator*. It is also called the *gradient* of the functional $E[u]$. When C is a linear operator we can choose $\eta(\lambda) = \lambda u$ and eq. (1.4.1) reduces to

$$E[u] = \int_0^1 \langle C\lambda u \,,\, u\,d\lambda\rangle = \frac{1}{2}\langle u \,,\, Cu\rangle \tag{1.4.2}$$

a well known result. The link between C and E is reinforced by

THEOREM 13. *If* C *is a monotone (resp. strictly monotone) operator, the potential* $E[u]$ *is a convex (resp. strictly convex) functional and viceversa*:

$$E[\lambda u' + (1-\lambda)\, u''] \leq \lambda E[u'] + (1-\lambda)\, E[u''] \qquad (\text{resp.} <). \tag{1.4.3}$$

For the proof see [1; Theorem 1.2].

THEOREM 14: (VARIATIONAL FORMULATION IN THE NONLINEAR CASE). *The solution of the fundamental equation (with* $\sigma = 0$*) makes stationary the functional* $S[\varphi] = E[D(\varphi)]$ *being* $E[u]$ *given by eq.* (1.4.1.).

Proof.

$$\begin{aligned} \delta_\varphi S[\varphi] &= \delta_\varphi E[D\varphi] = \langle CD(\varphi) \,,\, \delta D(\varphi)\rangle = \langle CD\varphi \,,\, D'_\varphi \delta\varphi\rangle = \\ &= \langle \widetilde{D}'_\varphi CD(\varphi) \,,\, \delta\varphi\rangle = 0\,. \end{aligned} \tag{1.4.4}$$

$S[\varphi]$ will be called the *action functional.*

THEOREM 15: *If* C *is invertible mapping the inverse operator* C^{-1} *is also of potential kind.*

Proof. It suffices to show that the elementary circulation

$$\langle \delta v \,,\, C^{-1}(v)\rangle \tag{1.4.5}$$

is the variation of a functional (automatically the circulation does not depend the line connecting two points).

From the identity

$$\langle \delta v \,,\, u\rangle \equiv \delta\langle v \,,\, u\rangle - \langle v \,,\, \delta u\rangle \tag{1.4.6}$$

it follows

$$\begin{aligned} \langle \delta v \,,\, C^{-1}(v)\rangle &\equiv \delta\langle v \,,\, C^{-1}(v)\rangle - \delta E[C^{-1}(v)] = \\ &= \delta\{\langle v, C^{-1}(v)\rangle - E[C^{-1}(v)]\} = \delta\overline{E}[v]\,. \end{aligned} \tag{1.4.7}$$

The new functional

$$\overline{E}[v] \overset{\text{def}}{=} [v \,,\, C^{-1}(v)] - E[C^{-1}(v)] \tag{1.4.8}$$

will be called the *dual potential.* The transform (1.4.8) is known as Legendre transform.

Combining Theorem 13 with this result we can state the

THEOREM 16: *If* C *is strictly monotone then the dual potential* $\overline{E}[v]$ *is convex.*

1.5. Dual balance equation

If we look at definition equation $u = D(\varphi)$ as an equation in which u is assigned and φ must be found we are faced with *compatibility conditions* on u (that are existence conditions for φ). If these conditions are found, be they $R(u) = 0$ we shall call the operator R an *annichilator* of D because $RD(\varphi) \equiv 0$.

This means that null manifold of R contains the range of D i.e. $\mathfrak{N}(R) \supseteq \mathfrak{R}(D)$.

If *all* elements u_0 for which $Ru_0 = 0$ can be cast into the form $u_0 = D\varphi$ then we call R a *minimal annichilator* because its null manifold coincides with the range of $D : \mathfrak{N}(R) = \mathfrak{R}(D)$.

In this case the compatibility condition $Ru = 0$ is not only necessary but also sufficient to assure that the equation $u = D\varphi$ admits a solution.

While the domain of R lies in the U–space, its range lies in another function space we choose *linear* and that we shall denote with T and call *dual source space*.

If definition equation is of the form $u = u_0 + D(\varphi)$ then the compatibility condition is $R(u - u_0) = 0$. If D and R are linear, this equation can be written $Ru = \tau$. The "incompatibility" term τ that can be viewed as a dual source variable. The equation $Ru = \tau$ is then called dual *balance equation*. Alongside the linear T–space we are lead to introduce another linear function space whose elements are of the same tensorial order as those of T. This space will be denoted with Ψ and called *dual configuration space*. These two spaces are put in duality introducing the bilinear functional denoted with $\langle \psi , \tau \rangle$. The space Ψ and the bilinear map $\langle \psi , \tau \rangle$ will be chosen so that the duality be separating and both spaces will be equipped with topologies that make the bilinear functional $\langle \psi , \tau \rangle$ continuous.

1.6. Relation between the dual balance and the dual definition operator (linear case)

With the bilinear form we can define the adjoint of the operator when the last is *linear* and when its domain is *dense* in the U–space

$$\langle \psi , Ru \rangle = \langle \tilde{R}\psi , u \rangle . \tag{1.6.1}$$

Now we can easily see that *the equation* $v = R\psi$ *gives a solution of the homogeneous balance equation* $\tilde{D}v = 0$.

We have in fact the following

THEOREM 17: *If* R *is a linear operator with domain dense in the* U–*space and range in* T, *that be an annichilator of* D, *then* $\tilde{D}$ *is an annichilator of* $\tilde{R}$.

Proof.

$$\langle \psi , RD\varphi \rangle = \langle \tilde{R}\psi , D\varphi \rangle = \langle \tilde{D}\tilde{R}\psi , \varphi \rangle . \tag{1.6.2}$$

Now if $\varphi \in \mathfrak{D}(\mathrm{D})$ $\mathrm{RD}\varphi = 0$ because R is an annichilator of D: then $\langle \psi , \mathrm{RD}\varphi \rangle = 0$ for every $\psi \in \Psi$. In particular this is true if $\psi \in \mathfrak{D}(\tilde{\mathrm{R}})$ then from $\langle \tilde{\mathrm{D}}\tilde{\mathrm{R}}\psi , \varphi \rangle = 0$ being $\varphi \in \mathfrak{D}(\mathrm{D})$ and $\overline{\mathfrak{D}}(\mathrm{D}) = \mathrm{U}$ follows $\tilde{\mathrm{D}}\tilde{\mathrm{R}}\psi = 0$. Thus $\tilde{\mathrm{D}}$ is annichilator of $\tilde{\mathrm{R}}$.

An obvious question can be raised: is the solution $v = \tilde{\mathrm{R}}\psi$ *general*, i.e. such that all elements v_0 such that $\mathrm{D}v_0 = 0$ are of the form $v_0 = \tilde{\mathrm{R}}\psi$? This implies that $\mathfrak{R}(\tilde{\mathrm{R}}) = \mathfrak{N}(\tilde{\mathrm{D}})$. As we shall now see the answer is linked with the question: is the condition $\mathrm{R}u = 0$ *sufficient* to assure that $u = \mathrm{D}\varphi$? We have in fact the following

THEOREM 18: *Let* U *and* T *be two complete topological vector spaces. If* R *is a closed linear operator with domain dense in* U *and closed range in* T *that is a minimal annichilator of* D, *and if* D *is a closed linear operator with closed range then the operator* $\tilde{\mathrm{D}}$ *is a minimal annichilator of* $\tilde{\mathrm{R}}$.

Proof. The hypothesis that R be a minimal annichilator means $\mathfrak{N}(\mathrm{R}) = \mathfrak{R}(\mathrm{D})$. Then $\mathfrak{N}^{\perp}(\mathrm{R}) = \mathfrak{R}^{\perp}(\mathrm{D})$. But, on account of the general property

$$\mathfrak{N}^{\perp}(\mathrm{R}) = \overline{\mathfrak{R}}(\tilde{\mathrm{R}}) \quad \text{and} \quad \mathfrak{R}^{\perp}(\mathrm{D}) = \overline{\mathfrak{N}}(\tilde{\mathrm{D}})\,. \tag{1.6.3}$$

Then $\overline{\mathfrak{R}}(\tilde{\mathrm{R}}) = \overline{\mathfrak{N}}(\tilde{\mathrm{D}})$. Because $\tilde{\mathrm{D}}$ is a closed operator its null space is also closed [6] i.e. $\overline{\mathfrak{N}}(\tilde{\mathrm{D}}) = \mathfrak{N}(\tilde{\mathrm{D}})$. Because R is closed with closed range then also $\tilde{\mathrm{R}}$ has closed range [6] then $\overline{\mathfrak{R}}(\tilde{\mathrm{R}}) = \mathfrak{R}(\tilde{\mathrm{R}})$. It follows

$$\mathfrak{R}(\tilde{\mathrm{R}}) = \mathfrak{N}(\tilde{\mathrm{D}}) \tag{1.6.4}$$

then $\tilde{\mathrm{D}}$ is a minimal annichilator for $\tilde{\mathrm{R}}$.

From this Theorem it follows that under the conditions given in the Theorem the equation $v = \tilde{\mathrm{R}}\psi$ gives the *general solution* of the balance equation $\tilde{\mathrm{D}}v = 0$.

1.7. THE GENERALIZED THEOREM OF VIRTUAL WORKS

The principle of virtual works of mechanics is a formulation of equilibrium expressed as a link between source variables (the forces) and configuration variables (the position vectors). The actual dependence of sources from configuration, i.e. constitutive equations, does not enter into the principle. We now show that the principle can be restated as a Theorem valid in the mathematical model on account of the relation between definition and balance operators.

THEOREM 19 (GENERALIZED THEOREM OF VIRTUAL WORKS): *the balance equation is equivalent to the equation*

$$\langle v , \delta u \rangle = \langle \sigma , \delta\varphi \rangle \tag{1.7.1}$$

Proof.

$$\langle v , \delta u\rangle = \langle v , \delta \mathrm{D}(\varphi)\rangle = \langle v , \mathrm{D}'_\varphi \delta\varphi\rangle = \langle \widetilde{\mathrm{D}}'_\varphi v , \delta\varphi\rangle = \langle \sigma , \delta\varphi\rangle . \tag{1.7.2}$$

COROLLARY 19–bis: *if* D *is a linear operator, balance equation is equivalent to the equation*

$$\langle v , u\rangle = \langle \sigma , \varphi\rangle \tag{1.7.3}$$

THEOREM 20 (DUAL GENERALIZED THEOREM OF VIRTUAL WORKS). *If the annichilator* R *is linear then the dual balance equation is equivalent to the equation*

$$\langle \delta\nu , u\rangle = \langle \delta\psi , \tau\rangle . \tag{1.7.4}$$

Proof.

$$\langle \delta\psi , \tau\rangle = \langle \delta\psi , \mathrm{R}u\rangle = \langle \widetilde{\mathrm{R}}\delta\psi , u\rangle = \langle \delta\widetilde{\mathrm{R}}\psi , u\rangle = \langle \delta v , u\rangle . \tag{1.7.5}$$

1.8. THE DUAL SCHEME

In order to relate dual source variables with dual configuration variables we need a mapping $\mathrm{V} \mapsto \mathrm{U}$. When the constitutive mapping C can be inverted then its inverse C^{-1} realizes the mapping $\mathrm{V} \mapsto \mathrm{U}$. When this happens we can consider the dual scheme $\psi \to v \to u \to \tau$. The mapping $\mathrm{RC}^{-1}\widetilde{\mathrm{R}}\psi = \tau$ will be called the dual fundamental mapping.

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