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Concentration functions. II

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Calcolo delle probabilità. — *Concentration functions*. II (*). Nota di WALTER HENGARTNER e RADU THEODORESCU, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Alcuni problemi di convergenza possono venire trattati con l'uso di numeri reali associati alle misure di probabilità; questi sono chiamati concentrazioni. In tal guisa gli AA. pervengono alle proprietà delle funzioni di concentrazione ottenute da P. Lévy [5], ed a generalizzare certi risultati più recenti di K. Ito [4].

Certain convergence problems may be handled by using real numbers associated with probability measures; these numbers are called concentrations. They basically yield, for these problems, the same properties as the concentration functions of P. Lévy [5]. In what follows we shall extend certain results due to K. Ito [4].

1. Consider the real line R and the σ–algebra $\mathfrak{B}$ of its Borel sets. Suppose that λ is a probability measure on $\mathfrak{B}$, absolutely continuous on R with respect to the Lebesgue measure m, and such that the Radon–Nykodim derivative $d\lambda/dm > 0$ on $(0, \alpha)$, $\alpha > 0$. Let h be a continuous mapping from $[0, 1]$ to $[0, 1]$ (1) satisfying the following conditions:

(h_1) for any sequence of characteristic functions $(\psi_n)_{n \in N^*}$, $N^* = \{1, 2, \cdots\}$, we have $\int_R h(|\psi_n(t)|)\, d\lambda(t) \to 1$ or zero iff $\int_R |\psi_n(t)|\, d\lambda(t) \to 1$ or zero respectively;

(h_2) $$h(0) = 0 \quad , \quad h(1) = 1 \text{ (2)}.$$

Examples of such functions are $h(x) = x^p$, $p > 0$, and any h such that $x^p \leq h(x) \leq x^q$, $p, q > 0$.

DEFINITION. – *The real number*

$$q_\mu = \int_R h(|\psi_\mu(t)|)\, d\lambda(t),$$

where ψ_μ *is the characteristic function of the probability measure* μ, *is called the concentration of* μ *with respect to* λ *and* h.

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(**) Nella seduta dell'11 marzo 1972.
(1) Any bounded interval $[\alpha, \beta]$ may be, of course, considered.
(2) This assumption does not lead to any loss of generality.

2. The following Theorem summarizes the main properties of q_μ.

THEOREM. – *We have*

1) $0 < q_\mu \leq 1$;

2) $q_\mu = 1$ *iff* $\mu = \delta_{x_0}$, *where* δ_{x_0} *is the degenerate probability measure concentrated at* x_0;

3) $q_\mu = q_{\mu_a} = q_{\mu^s}$, *where* μ_a *and* μ^s *are defined by*

$$\mu_a(A) = \mu(A - a) \quad , \quad \mu^s(A) = \mu(-A)$$

for every $A \in \mathfrak{B}$, *and*

$$A \pm B = \{a \pm b : a \in A, b \in B\};$$

4) $q_{\mu * \nu} \leq \min(q_\mu, q_\nu)$, *provided that h is increasing* (*here* « * » *denotes convolution*);

5) $q_{\mu_n} \to 0$ *as* $n \to \infty$ *iff* $Q_{\mu_n}(l) \to 0$ *as* $n \to \infty$ *for every* $l > 0$ (*here* $Q_\mu(l) = \sup_{x \in R} \mu([0, l] + x)$ *is the concentration function of* μ);

6) $q_{\mu_n} \to 1$ *as* $n \to \infty$ *iff* μ_n *is weakly essentially convergent to* δ_0 *as* $n \to \infty$;

7) $q_{\overset{n}{\underset{k=1}{*}} \mu_k} \to q > 0$ *as* $n \to \infty$ *iff* $\overset{n}{\underset{k=1}{*}} \mu_k$ *is weakly essentially convergent.*

Proof. – 1) There is always a neighbourhood of zero in which $|\psi_\mu(t)| > 1 - \varepsilon$, $\varepsilon > 0$, and a neighbourhood of zero in which $h(|\psi_\mu(t)|) > 1/2$; hence $\int_R h(|\psi_\mu(t)|)\, d\lambda(t) > 0$, so that $q_\mu > 0$. Next, it is easily seen that $q_\mu \leq 1$.

2) This property is mainly based on the fact that $|\psi_\mu(t)| \equiv 1$ m–a.s. in a neighbourhood of zero iff $\mu = \delta_{x_0}$.

3) and 4) are straightforward consequences of the elementary properties of characteristic functions.

5) Suppose that $q_{\mu_n} \to 0$ as $n \to \infty$. According to C. G. Esseen [1],

$$Q_{\mu_n}(L) \leq CL \int_{-2\pi/L}^{2\pi/L} |\psi_{\mu_n}(t)|\, dt,$$

where C is an absolute constant. Now let L large enough so that $2\pi/L < \alpha$; then we have

$$Q_{\mu_n}(L) \leq CL \int_{-\alpha}^{\alpha} |\psi_{\mu_n}(t)|\, dt. \tag{1}$$

But we know that $\int_R h(|\psi_{\mu_n}(t)|)\, d\lambda(t) \to 0$ as $n \to \infty$, and by (h_1) this is true iff $\int_R |\psi_{\mu_n}(t)|\, d\lambda(t) \to 0$ as $n \to \infty$; moreover, $\int_0^\alpha |\psi_{\mu_n}(t)|\, dt \to 0$ as $n \to \infty$

and, since $|\psi_{\mu_n}(-t)| = |\psi_{\mu_n}(t)|$ for every $t \in R$, we conclude that $\int_{-\alpha}^{\alpha} |\psi_{\mu_n}(t)| \, dt \to 0$ as $n \to \infty$. Hence, by (1), we obtain $Q_{\mu_n}(l) \to 0$ as $n \to \infty$ for every $l > 0$.

Conversely, suppose that $Q_{\mu_n}(l) \to 0$ as $n \to \infty$ for every $l > 0$. Since λ is absolutely continuous with respect to m we have $\psi_\lambda(t) \to 0$ as $|t| \to \infty$ (cf. E. Lukacs [7], p. 19). For any $\varepsilon > 0$ there is a number $t_0 > 0$ such that $|\psi_\lambda(t)| < \varepsilon$ if $|t| \geq t_0$. Now $q_{\mu_n} \to 0$ as $n \to \infty$ iff $\int_R |\psi_{\mu_n}(t)|^2 \, d\lambda(t) \to 0$ as $n \to \infty$. But

$$\int_R |\psi_{\mu_n}(t)|^2 \, d\lambda(t) = \iint_{R \times R} \psi_\lambda(x-y) \, d\mu_n(x) \, d\mu_n(y) \leq$$

$$\leq \iint_{|x-y| \leq t_0} d\mu_n(x) \, d\mu_n(y) + \iint_{|x-y| > t_0} |\psi_\lambda(x-y)| \, d\mu_n(x) \, d\mu_n(y) \leq$$

$$\leq Q_{\mu_n}(t_0) + \varepsilon .$$

Therefore $\limsup q_{\mu_n} \leq \varepsilon$ for every $\varepsilon > 0$; hence $q_{\mu_n} \to 0$ as $n \to \infty$.

6) Suppose that $q_{\mu_n} \to 1$ as $n \to \infty$. Then $\int_R |\psi_{\mu_n}(t)| \, d\lambda(t) \to 1$ as $n \to \infty$; now, since λ is absolutely continuous on R with respect to m and $d\lambda/dm > 0$ in $(0, \alpha)$, we have $\int_0^\alpha (1 - |\psi_{\mu_n}(t)|) \, dt \to 0$ as $n \to \infty$; hence

$$\int_0^\alpha 1 - |\psi_{\mu_n}(t)|^2 \, dt \leq 2 \int_0^\alpha (1 - |\psi_{\mu_n}(t)|) \, dt \to 0$$

as $n \to \infty$. A result due to Ju. V. Linnik [6], p. 55–57, implies that Q_{μ_n} converges weakly as $n \to \infty$ to the degenerate distribution function H_0 with mass point at zero. It follows that there exist constants a_n, $n \in N^*$, such that μ_{n,a_n} is weakly convergent to δ_0 as $n \to \infty$. For let a_n be the median of μ_n, $n \in N^*$. Then for every $\varepsilon > 0$ and $l > 0$ choose $n_0 = n_0(l, \varepsilon)$ sufficiently large so that $Q_{\mu_n}(l) > 1 - \varepsilon > 1/2$ for every $n \geq n_0$. Now, for every $n \in N^*$ there is a number $x_n \in [a_n - l, a_n]$ such that $Q_{\mu_n}(l) = \mu_n([0, l] + x_n)$. Let A be an open set. If $0 \notin A$, then $\liminf \mu_{n,a_n}(A) \geq 0$; on the other hand, if $0 \in A$, there exists $[-l, l] \subset A$, $l > 0$, such that $\mu_{n,a_n}(A) \geq Q_{\mu_n}(l) > 1 - \varepsilon$ for $n \geq n_0$, therefore $\liminf \mu_{n,a_n}(A) \geq 1 - \varepsilon$ for every $\varepsilon > 0$. Hence μ_{n,a_n} is weakly convergent to δ_0 as $n \to \infty$, i.e., μ_n is weakly essentially convergent to δ_0 as $n \to \infty$.

Conversely, suppose that μ_n is weakly essentially convergent to δ_0 as $n \to \infty$. In other words, there exist constants a_n, $n \in N^*$, such that

$|\psi_{n,a_n}(t)| \to 1$ as $n \to \infty$ for every $t \in R$. Therefore $\int_R h(|\psi_{\mu_{n,a_n}}(t)|)\, d\lambda(t) \to 1$ as $n \to \infty$, so that it follows from 3) that $q_{\mu_n} \to 1$ as $n \to \infty$.

7) By 5) we have $q_{\overset{n}{\underset{k=1}{*}}\mu_k} \to 0$ as $n \to \infty$ iff $Q_{\overset{n}{\underset{k=1}{*}}\mu_k}(l) \to 0$ as $n \to \infty$ for every $l > 0$, and this is true iff $\overset{n}{\underset{k=1}{*}}\mu_k$ is weakly essentially convergent.

3. We note that the results given by K. Ito [4], p. 42–49 may be obtained by taking $h(x) = x^2$, $x \in [0, 1]$, and $d\lambda(t) = dt/(1+t^2)$, $t \in R$.

Suppose now that $h(x) = -\log x$, $x \in [0, 1]$, and $d\lambda(t) = dt/\alpha$ if $0 \le t \le \alpha$ and $d\lambda(t) = 0$ otherwise. This special h, considered by A. Ja. Hinčin [3], which leads to the so-called Hinčin functional (cf. Ju. V. Linnik [6], p. 54) does not satisfy conditions (h_1) and (h_2). The corresponding concentration q_μ has a meaning only for those μ's whose characteristic functions are different from zero m–a.s. on $(0, \alpha)$. This explains why this concentration yields only a part of the results given in the preceding Theorem.

4. Finally, we note that the above Theorem also holds in s–dimensional Euclidian space.

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