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Strongly lower semi continuous correspondences in Banach Spaces

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Analisi funzionale. — *Strongly lower semi continuous correspondences in Banach Spaces.* Nota di FREDDY DELBAEN [(*)], presentata [(**)] dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra che in uno spazio di Banach una corrispondenza a valori convessi debolmente compatti che sia contemporaneamente debolmente semi–continua inferiormente e fortemente semi–continua superiormente, è fortemente semi–continua inferiormente. La dimostrazione si basa essenzialmente su di un risultato di Lindenstrauss e Troyanski sui punti fortemente esposti di insiemi convessi.

Let T be a metric space and let B be a Banach space. A convex closed valued mapping $\psi : T \to 2^B$ is called a correspondence. ψ is called strongly upper semi continuous if $\forall t_0 \in T$, $\forall t_n \to t_0$, $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$ $\psi(t_n) \subset \psi(t_0) + S(\varepsilon)$ where $S(\varepsilon)$ denotes the closed ε–ball around zero. ψ is called strongly lower semi continuous if $\forall t_0 \in T$, $\forall t_n \to t_0$, $\forall x_0 \in \psi(t_0)$ $\exists x_n \in \psi(t_n)$ $x_n \to x_0$ in the norm topology of B. ψ is called weakly lower semi continuous if $\forall t_0 \in T$, $\forall t_n \to t_0$, $\forall x_0 \in \psi(t_0)$, $\exists x_n \in \psi(t_n)$ such that $x_n \to x_0$ $\sigma(B, B^*)$ i.e. weakly. The aim of this note is to prove that a weakly lower semi continuous strongly upper semi continuous weakly compact valued correspondence is also strongly lower semi continuous. More generally we have

THEOREM. *If* $\psi : T \to B$ *is a correspondence (i.e. convex closed valued mapping* $\psi : T \to 2^B$).

If 1) $\forall t\ \psi(t)$ *is* $\sigma(B, B^*)$ *compact.*

2) $\psi : T \to B$ *is strongly upper semi continuous.*

3) $\forall x^* \in B^*$, $\forall t_0 \in T$, $\forall t_n \to t_0$, $\forall x_0 \in \psi(t_0)$ $\exists x_n \in \psi(t_n)$ *such that* $x^*(x_n) \to x^*(x_0)$.

Then ψ *is strongly lower semi continuous.*

Proof. We recall that a point $x \in K$ where K is weakly compact convex subset of B is called strongly exposed if

1) $\exists x^* \in B^*$ such that $\forall y \in K$, $y \neq x$ we have $x^*(y) < x^*(x)$ and
2) if $x_n \in K$ and $x^*(x_n) \to x^*(x)$ then $\| x_n - x \| \to 0$.

The key point of our Theorem is the fact that a weakly compact convex subset K of a Banach space is the closed convex hull of its strongly exposed points (see Troyanski [1], p. 178, Corollary 7).

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(**) Nella seduta dell'11 marzo 1972.

Let now $t_0 \in T$ and let $t_n \to t_0$. Let $x \in \psi(t_0)$ be a strongly exposed point with support functional x^*. We may suppose $\| x^* \| = 1$.

Suppose that x cannot be strongly approximated by elements of $\psi(t_n)$. Then there exists a subsequence (denoted by the same index n) and $\varepsilon > 0$ such that

$$\text{(1)} \qquad \forall n \,, \forall x_n \in \psi(t_n) \qquad \text{we have} \quad \| x - x_n \| \geq \varepsilon \,.$$

Let $\varepsilon_n = \frac{\varepsilon}{4n}$. By strongly upper semi continuity we can find a subsequence (still denoted by the same index n) such that $\forall n \quad \psi(t_n) \subset \psi(t_0) + S(\varepsilon_n)$. By weak lower semi continuity there exist $x_n \in \psi(t_n)$ such that $x^*(x_n) \to x^*(x)$. $\forall n$ we choose $y_n \in \psi(t_0)$ such that $\| y_n - x_n \| \leq \varepsilon_n$.

We claim that $x^*(y_n) \to x^*(x)$. Indeed

$$\begin{aligned} | x^*(y_n) - x^*(x) | &\leq | x^*(y_n) - x^*(x_n) | + | x^*(x_n) - x^*(x) | \\ &\leq \varepsilon_n + | x^*(x_n) - x^*(x) | \to 0 \,. \end{aligned}$$

But then $\| y_n - x \| \to 0$ since x is strongly exposed and this in turn implies

$$\| x_n - x \| \leq \| x_n - y_n \| + \| y_n - x \| \leq \varepsilon_n + \| y_n - x \| \to 0 \,.$$

This is a contradiction to (1). Until now we only proved that a strongly exposed point can be approximated by elements of $\psi(t_n)$. Hence we also proved that each finite convex combination of such points can be approximated. By a diagonalization procedure we can prove that each element of the strong convex closure of strongly exposed points can be approximated, hence each element of $\psi(t_0)$.

Remark 1. The metrisability of T is not used, in fact the argument remains true for generalized sequences.

Remark 2. The proof also yields the additional result that if x is strongly exposed, if $x_n \in \psi(t_n)$

$$x^*(x_n) \to x^*(x) \qquad \text{then} \quad \| x_n - x \| \to 0 \,.$$

Remark 3. In (3) of the Theorem $\forall x^* \in B^*$ can be replaced by (3*) $\forall x^* \in D$ where D is a strongly dense subset of B*. A more elegant formulation of (3*) is then (3′) $\forall x^* \in D$ where D is a strongly dense subset of B*. The mapping $T \to \mathbf{R}$

$$t \to \min \langle x^* \,, \psi(t) \rangle$$

is upper semi continuous.

Remark 4. If B is reflexive then assumption (1) can be replaced by (1′) $\forall t \; \psi(t)$ is strongly closed and bounded. Indeed it follows from the reflexivity of B that a convex strongly closed and bounded set is $\sigma(B, B^*)$ compact.

Remark 5. One may ask whether a weakly upper semi continuous, strongly lower semi continuous correspondence is strongly upper semi continuous. The following example shows that this is only true if weakly convergent sequences are also strongly convergent. Let $x_n \to 0 \ \sigma(B, B^*)$ and $\|x_n\| = 1$. Let $\psi(x_n)$ be the segment joining 0 and x_n and $\psi(0) = 0$. Clearly $\psi(x_n)$ is weakly upper semi continuous and strongly lower semi continuous. It is certainly not strongly upper semi continuous.

BIBLIOGRAPHY

[1] TROYANSKI, *On locally uniform convex and differentiable norms in certain non separable Banach Spaces*, « Studia Mathematica », 37, 173–180 (1971).