Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On a unilateral problem for the Navier-Stokes equations. Nota I

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **52** (1972), n.3, p. 337–342.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_52_3_337_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1972.

Analisi matematica. — *On a unilateral problem for the Navier–Stokes equations*. Nota I di GIOVANNI PROUSE (*), presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si considera, per le equazioni di Navier–Stokes, un problema « unilatero » che, tradotto in una formulazione debole, dà luogo ad una disequazione di Navier–Stokes. Si enuncia un Teorema di unicità ed esistenza della soluzione di tale disequazione nel caso bidimensionale ed un Teorema di esistenza di una soluzione « molto debole » in tre dimensioni spaziali. Le dimostrazioni di tali Teoremi vengono date nella Nota II.

1 - Let Ω be a tube with initial section Γ_1, final section Γ_2 and wall Γ_3; we assume that the tube actually terminates at Γ_2 and consider the motion of a viscous incompressible fluid which enters through Γ_1 and flows out of Ω and into the atmosphere through Γ_2. This problem obviously arises very often in practical cases, whenever the free flow of a fluid through an orifice is considered.

The motion of the fluid (of unit density and viscosity μ) in Ω is governed by the equations of Navier–Stokes

$$(1.1)\qquad \begin{cases} \dfrac{\partial u_j}{\partial t} - \mu\Delta u_j + \displaystyle\sum_{k=1}^{3} u_k \frac{\partial u_j}{\partial x_k} + \frac{\partial p}{\partial x_j} = f_j & (j = 1, 2, 3) \\ \operatorname{div} \vec{u} = 0 & (x \in \Omega), \end{cases}$$

$\vec{u} = (u_1, u_2, u_3)$ being the velocity, p the pressure, $\vec{f} = (f_1, f_2, f_3)$ the external force. The problem we shall study corresponds to the following initial and boundary conditions:

a) $\vec{u}(x, 0) = \vec{v}(x) \qquad x \in \Omega,$

b) $\vec{u}(x, t) = 0 \qquad x \in \Gamma_3 \ , \ t > 0,$

c) $p(x,t) + \frac{1}{2}|\vec{u}(x,t)|^2 = \varphi(x,t) \ , \ |\vec{u}(x,t)| = |\vec{u}(x,t) \times \vec{\nu}| \qquad x \in \Gamma_1, \ t > 0,$

d) $p(x, t) = 0, \qquad x \in \Gamma_2 \ , \ t > 0,$

e) $|\vec{u}(x, t)| = |\vec{u}(x, t) \times \vec{\nu}| \ , \ \vec{u}(x, t) \times \vec{\nu} \geq 0 \qquad x \in \Gamma_2 \ , \ t > 0,$

where $\vec{\nu}$ denotes the outer normal to Γ.

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(**) Nella seduta dell'11 marzo 1972.

Condition *a*) assigns the value of the velocity at all points of Ω when $t = 0$.

Condition *b*) imposes (in accordance with the limit layer theory) that on the wall Γ_3 the velocity of the fluid vanishes.

By *c*) we assign on the initial section Γ_1 the value of the total energy of the fluid which, being the density $=1$, is given by $p(x, t) + \frac{1}{2} |\vec{u}(x, t)|^2$; moreover, we assume that the velocity at each point of Γ_1 is directed as the normal to Γ_1.

Conditions *d*) and *e*) refer to the final section Γ_2; they interpret the fact that, since the fluid flows out of Ω and into the atmosphere (where the pressure is assumed $= 0$) through Γ_2, the pressure at all points of Γ_2 is $= 0$ and the velocity has the same direction as the outer normal $\vec{\nu}$.

It is obvious that the initial function $\vec{v}(x)$ will necessarily have to satisfy the conditions

$$
(1.2) \qquad
\begin{array}{ll}
\operatorname{div} \vec{v}(x) = 0 & x \in \Omega\,, \\
|\vec{v}(x)| = |\vec{v}(x) \times \vec{\nu}| & x \in \Gamma_1\,, \\
|\vec{v}(x)| = \vec{v}(x) \times \vec{\nu} & x \in \Gamma_2\,, \\
\vec{v}(x) = 0 & x = \Gamma_3\,.
\end{array}
$$

Let us now introduce some functional spaces in which the solutions of our problem will be defined. In what follows we shall always assume that Ω satisfies the cone property and that Γ_1 and Γ_2 are plane sections, perpendicular to the x_1–axis.

We denote by $\mathfrak{M}$ the manifold of vectors $\vec{v}(x)$ indefinitely differentiable and with null divergence in Ω and such that $|\vec{v}(x)| = |\vec{v}(x) \times \vec{\nu}|$ on $\Gamma_1 \cup \Gamma_2$, $\vec{v}(x) = 0$ on Γ_3; moreover, if N^σ is the closure of $\mathfrak{M}$ in $H^\sigma(\Omega)$, we can set, by the definitions given,

$$
(1.3) \qquad
\begin{aligned}
(\vec{v}, \vec{w})_{N^0} &= (\vec{v}, \vec{w})_{L^2(\Omega)} = \int_\Omega \sum_{j=1}^{3} v_j(x)\, w_j(x)\, d\Omega\,, \\
(\vec{v}, \vec{w})_{N^1} &= (\vec{v}, \vec{w})_{\overset{\circ}{H}{}^1(\Omega)} = \int_\Omega \sum_{j,k=1}^{3} \frac{\partial v_j}{\partial x_k} \frac{\partial w_j}{\partial x_k}\, d\Omega\,.
\end{aligned}
$$

Let $D(A)$ be the set of elements $\vec{u} \in N^1$ such that the linear form $\vec{v} \to (\vec{u}, \vec{v})_{N^1}$ is continuous in the topology of N^0; it is then possible to define a linear, self-adjoint, positive operator A, from $D(A)$ to N^0, such that

$$
(1.4) \qquad (\vec{u}, \vec{v})_{N^1} = (A\vec{u}, \vec{v})_{N^0} \qquad \forall \vec{u} \in D(A)\,, \quad \vec{v} \in N^1.
$$

Let us denote by A^σ (σ real ≥ 0) the power of order σ of A and by V_σ the domain of $A^{\sigma/2}$; V_σ is a Hilbert space with scalar product defined by

$$(1.5) \qquad (\vec{u}, \vec{v})_{V_\sigma} = (A^{\sigma/2}\vec{u}, A^{\sigma/2}\vec{v})_{N^0} \text{ [1]}.$$

By the definitions given,

$$N^1 = V_1 \quad , \quad N^0 = V_0 \quad , \quad H^\sigma(\Omega) \supset V_\sigma$$

and the interpolation property

$$(1.6) \qquad [V_\alpha , V_\beta]_\theta = V_{\alpha(1-\theta)+\beta\theta}$$

holds.

Finally, we introduce the closed convex set $K \subset V_1$:

$$(1.7) \qquad K = \{ \vec{v}(x) \mid \vec{v} \in V_1 \ , \ \vec{v}(x) \times \vec{\nu} \geq 0 \quad \text{a.e. on } \Gamma_2 \}.$$

Observe that, setting

$$b(\vec{u}, \vec{v}, \vec{w}) = \int_\Omega \sum_{j,k=1}^{3} u_k \frac{\partial v_j}{\partial x_k} w_j \, d\Omega$$

and assuming that $\operatorname{div} \vec{u} = 0$, we have

$$(1.8) \qquad b(\vec{u}, \vec{v}, \vec{w}) = - b(\vec{u}, \vec{w}, \vec{v}) + \int_\Gamma (\vec{v} \times \vec{w})(\vec{u} \times \vec{\nu}) \, d\Gamma$$

and, consequently,

$$(1.9) \qquad b(\vec{u}, \vec{v}, \vec{v}) = \frac{1}{2} \int_\Gamma |\vec{v}|^2 (\vec{u} \times \vec{\nu}) \, d\Gamma .$$

Moreover, $\forall \vec{u} \in D(A)$, $\vec{h} \in V_1$,

$$(1.10) \qquad (-\Delta\vec{u}, \vec{h})_{V_0} = (\vec{u}, \vec{h})_{V_1} - \int_\Gamma \sum_{j,k=1}^{3} \frac{\partial u_j}{\partial x_k} h_j \cos \nu x_k \, d\Gamma =$$

$$= (\vec{u}, \vec{h})_{V_1} - \int_\Gamma \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} h_j \, d\Gamma = (\vec{u}, \vec{h})_{V_1},$$

$$(1.11) \qquad \int_\Omega \sum_{j=1}^{3} \frac{\partial p}{\partial x_j} h_j \, d\Omega = - \int_\Omega p \sum_{j=1}^{3} \frac{\partial h_j}{\partial x_j} \, d\Omega + \int_\Gamma p\vec{h} \times \vec{\nu} \, d\Gamma .$$

(1) Identifying V_0 with its dual space, we shall denote by V'_σ the dual of V_σ.

We now associate to problem *b*), *c*), *d*), *e*) for system (1.1) the following weak formulation. A function $\vec{u}(t) = \{\vec{u}(x, t); x \in \Omega\}$ will be called a *weak solution* in $[0, T]$ of the Navier–Stokes equations (1.1) satisfying the boundary conditions *b*), *c*), *d*), *e*) if:

I_A) $\vec{u}(t) \in L^2(0, T; V_1)$, $\vec{u}'(t) \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$, $\vec{u}(t) \in K \quad \forall t \in [0, T]$;

II_A) $u(t)$ *satisfies the inequality*

$$(1.12) \quad \int_0^T \Big\{ (\vec{u}'(t), \vec{u}(t) - \vec{h}(t))_{V_0} + \mu(\vec{u}(t), \vec{u}(t) - \vec{h}(t))_{V_1} + b(\vec{u}(t), \vec{u}(t), \vec{u}(t) - \vec{h}(t)) - (\vec{f}(t), \vec{u}(t) - \vec{h}(t))_{V_0} + \int_{\Gamma_1} \Big[\varphi(x, t) - \frac{1}{2}|\vec{u}(x, t)|^2\Big] (\vec{u}(x, t) - \vec{h}(x, t)) \times \vec{\nu}\, d\Gamma_1 \Big\} dt \leq 0$$

$\forall \vec{h}(t) \in L^2(0, T; K)$.

Let us show that I_A), II_A) actually represent a weak formulation of the problem considered.

Observe, first of all, that condition I_A) implies that $\vec{u}(t)$ satisfies *e*) and the second of *c*). Moreover, if we set in (1.12) $\vec{h}(t) = 0$ and $\vec{h}(t) = 2\vec{u}(t)$ we obtain that necessarily

$$(1.13) \quad \int_0^T \Big\{ (\vec{u}'(t), \vec{u}(t))_{V_0} + \mu(\vec{u}(t), \vec{u}(t))_{V_1} + b(\vec{u}(t), \vec{u}(t), \vec{u}(t)) - (\vec{f}(t), \vec{u}(t))_{V_0} + \int_{\Gamma_1} \Big[\varphi(x, t) - \frac{1}{2}|\vec{u}(x, t)|^2\Big] \vec{u}(x, t) \times \vec{\nu}\, d\Gamma_1 \Big\} dt = 0 .$$

Hence, by (1.12), (1.13),

$$(1.14) \quad \int_0^T \Big\{ (\vec{u}'(t), \vec{h}(t))_{V_0} + \mu(\vec{u}(t), \vec{h}(t))_{V_1} + b(\vec{u}(t), \vec{u}(t), \vec{h}(t)) - (\vec{f}(t), \vec{h}(t))_{V_0} + \int_{\Gamma_1} \Big[\varphi(x, t) - \frac{1}{2}|\vec{u}(x, t)|^2\Big] \vec{h}(x, t) \times \vec{\nu}\, d\Gamma_1 \Big\} dt \geq 0$$

$\forall \vec{h}(t) \in L^2(0, T; K)$.

Let $\vec{l}(t)$ be a function $\in L^2(0, T; V_1)$, with $\vec{l}(x, t) = 0$ a.e. on $\Gamma_2 \times [0, T]$; setting in (1.14) $\vec{h}(t) = \pm \vec{l}(t)$, it follows that

$$(1.15) \quad \int_0^T \Big\{ (\vec{u}'(t), \vec{l}(t))_{V_0} + \mu (\vec{u}(t), \vec{l}(t))_{V_1} + b(\vec{u}(t), \vec{u}(t), \vec{l}(t)) - (\vec{f}(t), \vec{l}(t))_{V_0} + $$
$$+ \int_{\Gamma_1} \left[\varphi(x, t) - \frac{1}{2} |\vec{u}(x, t)|^2 \right] \vec{l}(x, t) \times \vec{\nu} \, d\Gamma_1 \Big\} dt = 0 .$$

The same equation (1.15) can also be obtained multiplying by l_j the first of (1.1), adding, integrating over $\Omega \times [0, T]$ and bearing in mind the second of (1.1), (1.10), (1.11) and conditions *b*), *c*), *d*). Hence, if $\vec{u}(t)$ satisfies II_A) (and consequently (1.15)), it is a weak solution in $\Omega \times [0, T]$ of the Navier–Stokes equations satisfying conditions *b*), *d*) and the first of *c*).

In § 2 we shall prove that, in the case of two space dimensions (i.e. for plane flows), there exists, under appropriate assumptions on the data, a unique weak solution in $\Omega \times [0, T]$ of system (1.1) satisfying the initial and boundary conditions *a*), *b*), *c*), *d*), *e*).

In order to study the general three–dimensional case, we introduce a " very weak " formulation of the problem considered. Observe, for this, that, by (1.9) and conditions *b*), *e*),

$$(1.16) \qquad b(\vec{u}, \vec{u}, \vec{u}) = \frac{1}{2} \int_{\Gamma_1 \cup \Gamma_2} |\vec{u}|^2 (\vec{u} \times \vec{\nu}) \, d\Gamma \geq \frac{1}{2} \int_{\Gamma_1} |\vec{u}|^2 (\vec{u} \times \vec{\nu}) \, d\Gamma ;$$

moreover,

$$(1.17) \quad (\vec{u}'(t), \vec{u}(t) - \vec{h}(t))_{V_0} = (\vec{h}'(t), \vec{u}(t) - \vec{h}(t))_{V_0} + (\vec{u}'(t) - \vec{h}'(t), \vec{u}(t) - \vec{h}(t))_{V_0} =$$
$$= \frac{1}{2} \frac{d}{dt} \|\vec{u}(t) - \vec{h}(t)\|^2_{V_0} + (\vec{h}'(t), \vec{u}(t) - \vec{h}(t))_{V_0} .$$

Let us assume that $\vec{h}(t)$, besides satisfying the assumptions made so far, is such that $\vec{h}'(t) \in L^2(0, T; V_0)$, $\vec{h}(0) = \vec{u}(0)$. Substituting (1.16), (1.17) in (1.12), we obtain, denoting by $\langle , \rangle$ the duality between V_1' and V_1,

$$(1.18) \quad \int_0^T \Big\{ (\vec{h}'(t), \vec{u}(t) - \vec{h}(t))_{V_0} + \mu (\vec{u}(t), \vec{u}(t) - \vec{h}(t))_{V_1} - b(\vec{u}(t), \vec{u}(t), \vec{h}(t)) -$$
$$- \langle \vec{f}(t), \vec{u}(t) - \vec{h}(t) \rangle + \int_{\Gamma_1} \varphi(x, t) (\vec{u}(x, t) - \vec{h}(x, t)) \times \vec{\nu} \, d\Gamma_1 +$$
$$+ \frac{1}{2} \int_{\Gamma_1} |\vec{u}(x, t)|^2 \, \vec{h}(x, t) \times \vec{\nu} \, d\Gamma_1 \Big\} dt \leq 0 .$$

We shall therefore say that $\vec{u}(t)$ is a *very weak* solution in $[0, T]$ of the Navier–Stokes equations satisfying the initial and boundary conditions *a*), *b*), *c*), *d*), *e*), if

$\mathrm{I_B}$) $\vec{u}(t) \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$, $\vec{u}(t) \in K$ *a.e. on* $[0, T]$;

$\mathrm{II_B}$) $\vec{u}(t)$ *satisfies* (1.18) $\forall \vec{h}(t) \in L^2(0, T; K)$, *with* $\vec{h}'(t) \in L^2(0, T; V_0)$, $\vec{h}(0) = \vec{v}$.

In § 3 we shall prove an existence Theorem on $[0, T]$ of a very weak solution in the general three–dimensional case. We obtain therefore results similar to the well known ones which have been proved for the classical mixed problem with Dirichlet boundary conditions: when the space dimensions are 2, an existence and uniqueness Theorem holds for an appropriately defined weak solution, whereas, for more than two dimensions, the existence of a solution is proved in a functional class in which the uniqueness is not guaranteed.

Let us now state the two Theorems that, as mentioned above, will be proved in the next two paragraphs.

THEOREM 1. – *Assume that the fluid is initially at rest, i.e. that* $\vec{v}(x) = 0$, $\varphi(x, 0) = 0$ *and that* $\vec{f}(t) \in L^2(0, T; V_0)$, $\vec{f}'(t) \in L^2(0, T; V_0)$, $\varphi(t) \in L^2(0, T; L^2(\Gamma_1))$, $\varphi'(t) \in L^2(0, T; L^2(\Gamma_1))$. *There exists then in* $[0, T]$ *one and only one weak solution of the two-dimensional problem.*

THEOREM 2. – *Assume that* $\vec{v}(x) \in K$, $\vec{f}(t) \in L^2(0, T; V_1')$, $\varphi(t) \in L^2(0, T; L^2(\Gamma_1))$. *There exists then, in the three-dimensional case, at least one very weak solution in* $[0, T]$.