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Product of Composition in tensor products of bimodules

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Matematica. — *Product of Composition in tensor products of bimodules* (*). Nota di SERGE VASILACH, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Definizione e studio delle proprietà del prodotto di composizione nei prodotti tensoriali di famiglie finite di bimoduli. Si determinano le condizioni affinché questa operazione di composizione risulti chiusa nei prodotti tensoriali considerati. Ciò permette di costruire delle algebre di composizione che verranno poi utilizzate nella risoluzione con metodi algebrici di equazioni differenziali ordinarie od a derivate parziali a coefficienti variabili.

1. PRELIMINARIES

Let: $\mathbf{Z}$ be the ring of rational integers: i.e. $\mathbf{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$. Let $(A_j)_{1 \le j \le m}$ be a family of m arbitrary rings, with unit element 1. Let $(E_j)_{1 \le j \le m}$ be a family of m bimodules satisfying the following conditions:

a) $\quad E_1$ is a (A_m, A_1)–bimodule;

and

b) $\quad E_j$ is a (A_{j-1}, A_j)–bimodule.

Let $(F_j)_{1 \le j \le m}$, $(G_j)_{1 \le j \le m}$ be two families of m bimodules satisfying the same conditions *a*), *b*). Let $(F_j^*)_{1 \le j \le m}$ be the family of algebraic duals of the F_j, satisfying the following conditions:

*a**) $\quad F_1^*$ is a (A_1, A_m)–bimodule;

*b**) $\quad F_j^*$ is a (A_j, A_{j-1})–bimodule, $\quad \forall j \in [2, m] \subset \mathbf{N}$.

For each (A, B)–bimodule E considered as left A-module, let $\langle x, x^* \rangle$ be the element $x^*(x)$ of A, $\forall x \in E$, $\forall x^* \in E^*$, satisfying the conditions (cf. Bourbaki [1], § 2, n. 3):

$$\langle x + y , x^* \rangle = \langle x , x^* \rangle + \langle y , x^* \rangle$$

$$\langle x , x^* + y^* \rangle = \langle x , x^* \rangle + \langle x , y^* \rangle$$

$$\langle \alpha x , x^* \rangle = \alpha \langle x , x^* \rangle$$

$$\langle x , x^* \alpha \rangle = \langle x , x^* \rangle \alpha$$

for x, y in E, x^*, y^* in E^*, and $\alpha \in A$.

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Likewise, if we consider the (A , B)–bimodule E, as right B–module, we design by $\langle x^*, x\rangle$ the value $x^*(x)$ of the linear functional $x \in E^*$ where E^* is a left bimodule; then we have:

$$x^*(x) = \langle x^*, x\rangle$$

$$\langle x^*, x\beta\rangle = \langle x^*, x\rangle \beta$$

$$\langle \beta x^*, x\rangle = \beta \langle x^*, x\rangle$$

for $x \in E$, $x \in E^*$, $\beta \in B$.

2. Tensor product of bi–modules

We consider the tensor product of bimodules defined as follows: (cf. Bourbaki [1], § 3, n. 9):

$$\text{(1)} \qquad \bigotimes_{j=1}^{m} E_j = {}_{A_m}E_1 \otimes_{A_1} E_2 \otimes_{A_2} \cdots \otimes_{A_{m-1}} E_{m_{A_1}},$$

$$\text{(2)} \qquad \bigotimes_{j=1}^{m} F_j = {}_{A_m}F_1 \otimes_{A_1} F_2 \otimes_{A_2} \cdots \otimes_{A_{m-1}} F_{m_{A_1}},$$

$$\text{(3)} \qquad \bigotimes_{j=1}^{m} F^*_{m-j+1} = {}_{A_m}F^*_m \otimes_{A_{m-1}} F^*_{m-1} \otimes_{A_{m-2}} F^*_{m-2} \otimes \cdots \otimes_{A_1} F^*_{1_{A_m}},$$

$$\text{(4)} \qquad \bigotimes_{j=1}^{m} G_j = {}_{A_m}G_1 \otimes_{A_1} G_2 \otimes_{A_2} \cdots \otimes_{A_{m-1}} G_{m_{A_1}}.$$

This definition shows that all tensor products are **Z**–modules (cf. [1], § 3).

3. Product of composition of tensor products of bimodules

Let:

$$a = \bigotimes_{j=1}^{m} a_j \in \bigotimes_{j=1}^{m} E_j \quad ; \quad b = \bigotimes_{j=1}^{m} b_j \in \bigotimes_{j=1}^{m} F_j ;$$

$$c^* = \bigotimes_{j=1}^{m} c^*_{m-j+1} \in \bigotimes_{j=1}^{m} F_{m-j+1} \quad ; \quad d = \bigotimes_{j=1}^{m} d_j \in \bigotimes_{j=1}^{m} G_j$$

be arbitrary elements.

Definition 1. *We call right product of composition of*

$$a \otimes b \in \left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} F_j\right)$$

and

$$c^* \otimes d \in \left(\bigotimes_{j=1}^{m} F^*_{m-j+1}\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} G_j\right),$$

the **Z**–*bilinear mapping*:

$$\left(\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} F_j\right)\right) \times \left(\left(\bigotimes_{j=1}^{m} F_{m-j+1}\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} G_j\right)\right)$$

into $\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} G_j\right)$, *defined by*

$$(a \otimes b\,,\, c^* \otimes d) \mapsto \left(\bigotimes_{j=1}^{m} (a_j \langle c_j^*, b_j\rangle) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} d_j\right)\right), \tag{5}$$

where: $\langle c_j^*, b_j\rangle \in A_j$, *for* $1 \leq j \leq m$

and

$$\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} G_j\right) = E_1 \otimes_{A_1} E_2 \otimes_{A_2} \cdots \otimes_{A_{m-1}} E_m \otimes_{A_m} G_1 \otimes_{A_1} G_2 \otimes \cdots \otimes_{A_{m-1}} G_m .$$

Analogous notations for

$$\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} F_j\right) \quad \text{and} \quad \left(\bigotimes_{j=1}^{m} F^*_{m-j+1}\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} G_j\right).$$

DEFINITION 2. *We call left product of composition of*

$$a \otimes b \in \left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} F_j\right) \quad \textit{and} \quad c^* \otimes d \in \left(\bigotimes_{j=1}^{m} F^*_{m-j+1}\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} G_j\right),$$

the **Z**–*bilinear mapping of* $\left(\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} F_j\right)\right) \times \left(\left(\bigotimes_{j=1}^{m} F^*_{m-j+1}\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} G_j\right)\right)$, *into* $\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} G_j\right)$ *defined by*

$$(a \otimes b\,,\, c^* \otimes d) \mapsto \left(\bigotimes_{j=1}^{m} a_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} \langle b_j\,,\, c_j^*\rangle\, d_j\right) \tag{6}$$

where:

$$\langle b_1\,,\, c_1^*\rangle \in A_m \quad \textit{and} \quad \langle b_j\,,\, c_j^*\rangle \in A_{j-1}, \qquad \textit{for} \quad 2 \leq j \leq m. \tag{7}$$

If for

$$u = a \otimes b \quad , \quad v = c^* \otimes d \tag{8}$$

we note by

$$(u \circ v)_d \quad (\text{resp. } (u \circ v)_g)$$

the right (resp. left) product of composition given by the Definition 1 (resp. Def. 2), we have:

$$(u \circ v)_d = \left(\bigotimes_{j=1}^{m} (a_j \langle c_j^*, b_j\rangle)\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} d_j\right) \tag{9}$$

where $\langle c_j^*, b_j\rangle \in A_j$, for $1 \leq j \leq m$, and

$$(u \circ v)_g = \left(\bigotimes_{j=1}^{m} a_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} \langle b_j, c_j^*\rangle d_j\right) \tag{10}$$

with $\langle b_1, c_1^*\rangle \in A_m$ and $\langle b_j, c_j^*\rangle \in A_{j-1}$, for $2 \leq j \leq m$.

Remark 1. In particular, if the rings $(A_j)_{1\leq j\leq m}$ are identical to the same ring A and if A is commutative, then (cf. Bourbaki [1], § 3, n. 9) the formulas (1), (2), (3), (4) of n. 2 above take the form

$$\bigotimes_{j=1}^{m} E_j = E_1 \otimes_A E_2 \otimes \cdots \otimes_A E_m = E_1 \otimes E_2 \otimes \cdots \otimes E_m \tag{11}$$

with analogous expressions for $\bigotimes_{j=1}^{m} F_j$, $\bigotimes_{j=1}^{m} F_{m-j+1}^*$ and $\bigotimes_{j=1}^{m} G_j$. Under these conditions the right and left products of composition defined by (9) and (10) coincide and we may write:

$$u \circ v = a\langle b, c^*\rangle \otimes d = a \otimes \langle b, c^*\rangle d = \langle b, c^*\rangle (a \otimes d), \tag{12}$$

where $a = \bigotimes_{j=1}^{m} a_j \in \bigotimes_{j=1}^{m} E_j$; $b = \bigotimes_{j=1}^{m} b_j \in \bigotimes_{j=1}^{m} F_j$,

$$c^* = \bigotimes_{j=1}^{m} c_j^* \quad , \quad d = \bigotimes_{j=1}^{m} d_j \tag{13}$$

and

$$\langle b, c^*\rangle = \langle c^*, b\rangle = \prod_{j=1}^{m} \langle b_j, c_j^*\rangle = \prod_{j=1}^{m} \langle c_j^*, b_j\rangle. \tag{14}$$

In this case $\bigotimes_{j=1}^{m} E_j$, $\bigotimes_{j=1}^{m} F_j$, $\bigotimes_{j=1}^{m} F_j^*$, $\bigotimes_{j=1}^{m} G_j$ are A–modules, and if A is a commutative field, then there are vector spaces over A.

4. Associativity of the composition product

Consider: $\bigotimes_{j=1}^{m} G_{m-j+1}^* = G_m^* \otimes_{A_{m-1}} G_{m-1}^* \otimes_{A_{m-2}} \cdots \otimes_{A_1} G_{1A_m}^*$ where G_j^* is the algebraic dual of G_j, $\forall j \in [1, m] \subset \mathbf{N}$. Let $(H_j)_{1\leq j\leq m}$ be a family of bimodules satisfying the conditions *a*), *b*), of n. 1. Let $(H_j^*)_{i\leq j\leq m}$ be the family of algebraic duals H_j^* of the H_j, for $j \in [1, m]$, the H_j^* satisfying the conditions *a**), *b**), of n. 1. Consider, on the other hand, the arbitrary elements g^*, h given by:

$$g^* = (g_m^* \otimes g_{m-1}^* \otimes \cdots g_1) \in \bigotimes_{j=1}^{m} G_{m-j+1}^*$$

$$h \otimes h_1 = h_2 \otimes \cdots \otimes h_m \in \bigotimes_{j=1}^{m} H_j = H_1 \otimes_{A_1} H_2 \otimes_{A_2} \cdots \otimes_{A_{m-1}} H_m.$$

If we set

$$w = g^* \circ h = (g_m^* \otimes g_{m-1}^* \otimes \cdots \otimes g_1^*) \otimes_{A_m} (h_1 \otimes h_2 \otimes \cdots \otimes h_m);$$

then from (9), we get

$$(15) \qquad ((u \circ v)_d \circ w)_d = \left[\left(\bigotimes_{j=1}^{m} (a_j \langle c_j^*, b_j\rangle) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} d_j\right)\right] \circ \left[\left(\bigotimes_{j=1}^{m} g_{m-j+1}^*\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} h_j\right)\right] =$$
$$= \bigotimes_{j=1}^{m} (a_j \langle c_j^*, b_j\rangle \langle g_j^*, d_j\rangle) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} h_j\right) \in \left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} H_j\right),$$

with $\langle c_j^*, b_j\rangle \langle g_j^*, d_j\rangle \in A_j$, for $j \in [1, m]$.

This operation of composition is not associative, since the right product of composition $(v \circ w)_d$ *is not defined, in general.* Indeed, if we carry out the product of composition in the same way as for $(u \circ v)_d$ we find:

$$(16) \qquad (v \circ w)_d = \left[\left(\bigotimes_{j=1}^{m} c_{m-j+1}^*\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} d_j\right)\right] \circ \left[\left(\bigotimes_{j=1}^{m} g_{m-j+1}^*\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} h_j\right)\right] =$$
$$= \left(\bigotimes_{j=1}^{m} c_{m-j+1}^* \langle g_j^*, d_j\rangle\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} h_j\right).$$

But

$$(17) \quad \begin{cases} c_m^* \in F_m^* = (A_m, A_{m-1})\text{–bimodule} & \text{for } j = 1, \\ c_{m-j+1}^* \in F_{m-j+1}^* = (A_{m-j+1}, A_{j-m})\text{–bimodule} & \text{for } 1 \leq j \leq m-1 \\ \text{and } \; c_1^* \in F_1^* = (A_1, A_m)\text{–bimodule}, & \end{cases}$$

On the other hand we have:

$$(18) \qquad \langle g_j^*, d_j\rangle \in A_j, \qquad \text{for } \; i \leq j \leq m.$$

The relations (17) and (18) show that the elements $c_{m-j+1}^* \langle g_j^*, d_j\rangle$ are not defined, in general, since c_{m-j+1}^* is an element of F_{m-j+1}^* which is a A_{m-j+1} right–module, while the $\langle g_j^*, d_j\rangle$ is an element of ring A_j, which, in general, is distinct of A_{m-j+1}. The composition product should be defined if

$$A_{m-j+1} = A_j, \qquad \forall j \in [1, m].$$

On the other hand, the product of composition $(v \circ w)_g$ is defined, since

$$(19) \qquad (v \circ w)_g = \left(\bigotimes_{j=1}^{m} c_{m-j+1}^*\right) \otimes \left(\bigotimes_{j=1}^{m} \langle g_j^*, d_j\rangle h_j\right)$$

where $\qquad \langle g_j^*, d_j\rangle \in A_j \qquad \forall j \in [1, m].$

Therefore $(u \circ (v \circ w)_g)_d$ is also defined since

$$(20) \qquad (u \circ (v \circ w)_g)_d = \left(\bigotimes_{j=1}^{m} a_j \langle c_j^*, b_j\rangle\right) \otimes \left(\bigotimes_{j=1}^{m} \langle g_j^*, d_j\rangle h_j\right),$$

with $(u \circ (v \circ w)_g)_d \in \left(\bigotimes_{j=1}^{m} E_j\right) \otimes \left(\bigotimes_{j=1}^{m} H_j\right)$, and $((u \circ v)_d \circ w)_d \neq (u \circ (v \circ w)_g)_d$.

But, if $A_{m-j+1} = A_j$, $\forall j \in [1, m]$, we have:

$$((u \circ v)_d \circ w)_d = (u \circ (v \circ w)_d)_d .$$

In particular, this is true, if the rings $(A_j)_{1 \leq j \leq m}$ are all equal to the same commutative ring A.

Analogous conclusions for the left composition products. Under these conditions, if A is a commutative ring (resp. field), then the product of composition, of tensor products of A–modules (resp. A–vector spaces) is associative, i.e.

$$u \circ (v \circ w) = (u \circ v) \circ w = u \circ v \circ w, \tag{21}$$

for any elements

$$u \in \left(\bigotimes_{j=1}^{m} E_j\right) \otimes \left(\bigotimes_{j=1}^{m} F_j\right),$$

$$v \in \left(\bigotimes_{j=1}^{m} F^*_{m-j+1}\right) \otimes \left(\bigotimes_{j=1}^{m} G_j\right)$$

and

$$w \in \left(\bigotimes_{j=1}^{m} G^*_{m-j+1}\right) \otimes \left(\bigotimes_{j=1}^{m} H_j\right).$$

5. THE RING OF COMPOSITION $\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} E^*_{m-j+1}\right)$

Let us return to the general case and let $(A_j)_{1 \leq j \leq m}$ be a finite family of m arbitrary rings with unit element, denoted by 1. Let $(E_j)_{1 \leq j \leq m}$ be a family of m bimodules satisfying the conditions a), b), of n. 1. Let $(E^*_{m-j+1})_{1 \leq j \leq m}$ be a family of m bimodules, algebraic duals of E_{m-j+1}, for $1 \leq j \leq m$, and satisfying the conditions a^*), b^*), of n. 1. Consider the right (resp. left) product of composition of two arbitrary elements

$$u = a \otimes b^* \quad \text{and} \quad v = c \otimes d \quad \text{of} \quad \left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} E^*_{m-j+1}\right).$$

From the formulas (9) and (10) we get:

$$(u \circ v)_d = \left(\bigotimes_{j=1}^{m} (a_j \langle b^*_j , c_j \rangle)\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} d^*_{m-j+1}\right) \tag{22}$$

where

$$\langle b^*_j , c_j \rangle \in A_j , \qquad \forall j \in [1, m].$$

The formula (22) shows that $(u \circ v)_d$ is an element of $\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} E^*_{m-j+1}\right)$.

On the other hand, if we set

$$w = \left(\bigotimes_{j=1}^{m} g_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} h^*_{m-j+1}\right), \tag{23}$$

we have:

$$(v \circ w)_d = \left(\bigotimes_{j=1}^{m} (c_j \langle d_j^*, g_j\rangle)\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} h_{m-j+1}^*\right). \tag{24}$$

This product of composition is defined by virtue of the conditions *a*), *b*) and *a**), *b**) which are satisfied by the bimodules $(E_j)_{1\le j\le m}$ and $(E_{m-j+1}^*)_{1\le j\le m}$ respectively. According to (24), we obtain:

$$(u \circ (v \circ w)_d)_d = \left(\bigotimes_{j=1}^{m} (a_j \langle b_j^*, c_j\rangle \langle d_j^*, g_j\rangle)\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} h_{m-j+1}^*\right). \tag{25}$$

Likewise, (22) and (23) give us

$$((u \circ v)_d \circ w)_d = \tag{26}$$

$$= \left[\left(\bigotimes_{j=1}^{m} (a_j \langle b_j^*, c_j\rangle)\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} d_{m-j+1}^*\right)\right] \otimes_{A_m} \left[\left(\bigotimes_{j=1}^{m} g_j\right) \otimes \left(\bigotimes_{j=1}^{m} h_{m-j+1}^*\right)\right] =$$

$$= \left[\bigotimes_{j=1}^{m} (a_j \langle b_j^*, c_i\rangle \langle d_j^*, g_j\rangle)\right] \otimes_{A_m} \left(\bigotimes_{j=1}^{m} h_{m-j+1}^*\right).$$

The formulas (25) and (26) show that the right product of composition of elements of $\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} E_{m-j+1}^*\right)$ is associative.

Therefore, we can write

$$((u \circ v)_d \circ w)_d = (u \circ (v \circ w)_d)_d = (u \circ v \circ w)_d. \tag{27}$$

For the left product of composition, we have:

$$(u \circ v)_g = \left[\left(\bigotimes_{j=1}^{m} a_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} b_{m-j+1}^*\right)\right] \circ \left[\left(\bigotimes_{j=1}^{m} c_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} d_{m-j+1}^*\right)\right]$$

where

$$u = \left(\bigotimes_{j=1}^{m} a_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} b_{m-j+1}^*\right), \tag{28}$$

and

$$v = \left(\bigotimes_{j=1}^{m} c_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} d_{m-j+1}^*\right) \tag{29}$$

are two arbitrary elements of $\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} E_{m-j+1}^*\right)$.

Therefore

$$(u \circ v)_g = \left(\bigotimes_{j=1}^{m} a_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} (\langle c_{m-j+1}, b_{m-j+1}^*\rangle \, d_{m-j+1}^*)\right) \tag{30}$$

where

$$\langle c_j , b_j^* \rangle \in A_j , \qquad \forall j \in [1 , m] .$$

For v , w arbitrary elements of $\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} E_j^*\right)$ given by (29) and (23), we get

$$(31) \qquad (v \circ w)_g = \left(\bigotimes_{j=1}^{m} c_j\right) \otimes_{A_m} \left(\left(\bigotimes_{j=1}^{m} \langle g_{m-j+1} , d_{m-j+1}^* \rangle h_{m-j+1}^*\right)\right),$$

whence

$$((u \circ v)_g \circ w)_g = \left(\bigotimes_{j=1}^{m} a_j\right) \otimes_{A_m} \left(\left(\bigotimes_{j=1}^{m} \langle c_{m-j+1} , b_{m-j+1}^* \rangle \langle g_{m-j+1} , d_{m-j+1}^* \rangle h_{m-j+1}^*\right)\right)$$

and

$$(u \circ (v \circ w)_g)_g = \left[\left(\bigotimes_{j=1}^{m} a_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} b_{m-j+1}^*\right)\right] \circ$$

$$\circ \left[\left(\bigotimes_{j=1}^{m} c_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} \langle g_{m-j+1} , d_{m-j+1}^* \rangle h_{m-j+1}^*\right)\right] =$$

$$= \left(\bigotimes_{j=1}^{m} a_j\right) \otimes \left(\bigotimes_{j=1}^{m} (\langle c_{m-j+1} , b_{m-j+1}^* \rangle \langle g_{m-j+1} , d_{m-j+1} \rangle h_{m-j+1}^*)\right).$$

Therefore we have:

$$(32) \qquad ((u \circ v)_g \circ w)_g = (u \circ (v \circ w)_g)_g = (u \circ v \circ w)_g .$$

Consequently the right (resp. left) product of composition of elements of $\left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} E_{m-j+1}\right)$ is associative.

On the other hand, we have:

$$(33) \qquad (v \circ u)_d = \left[\left(\bigotimes_{j=1}^{m} c_j\right) \otimes \left(\bigotimes_{j=1}^{m} d_{m-j+1}^*\right)\right] \circ \left[\left(\bigotimes_{j=1}^{m} a_j\right) \otimes \left(\bigotimes_{j=1}^{m} b_{m-j+1}^*\right)\right] =$$

$$= \left(\bigotimes_{j=1}^{m} (c_j \langle d_j^* , a_j \rangle)\right) \otimes \left(\bigotimes_{j=1}^{m} b_{m-j+1}^*\right) \in \left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} E_{m-j+1}^*\right).$$

From (22) and (33) we get: $(u \circ v)_d \neq (v \circ u)_d$.

Likewise

$$(34) \qquad (v \circ u)_g = \left(\bigotimes_{j=1}^{m} c_j\right) \otimes \left(\bigotimes_{j=1}^{m} \langle a_{m-j+1} , d_{m-j+1}^* \rangle b_{m-j+1}^*\right) \in$$

$$\in \left(\bigotimes_{j=1}^{m} E_j\right) \otimes_{A_m} \left(\bigotimes_{j=1}^{m} E_{m-j+1}^*\right),$$

and (30) and (34) show that $(u \circ v)_g \neq (v \circ u)_g$.

Consider now the arbitrary elements

$$u_1 = \left(\bigotimes_{j=1}^{m} a_j^{(1)}\right) \otimes \left(\bigotimes_{j=1}^{m} b_{m-j+1}^{*(1)}\right) \quad , \quad v_1 = \left(\bigotimes_{j=1}^{m} c_j^{(1)}\right) \otimes \left(\bigotimes_{j=1}^{m} d_{m-j+1}^{*(1)}\right)$$

of $\left(\bigotimes_{j=1}^{m} \mathrm{E}_j\right) \otimes_{\mathrm{A}_m} \left(\bigotimes_{j=1}^{m} \mathrm{E}_{m-j+1}^{*}\right)$. We have:

$$\text{(35)} \qquad ((u + u_1) \circ v)_d = \left(\bigotimes_{j=1}^{m} (a_j \langle b_j^*, c_j\rangle)\right) \otimes \left(\bigotimes_{j=1}^{m} d_{m-j+1}^{*}\right) +$$
$$+ \left(\bigotimes_{j=1}^{m} (a_j^{(1)} \langle b_j^{*(1)}, c_j\rangle)\right) \otimes \left(\bigotimes_{j=1}^{m} d_{m-j+1}^{*}\right) = (u \circ v)_d + (u_1 \circ v)_d .$$

In the same way we obtain the relations:

$$\text{(36)} \qquad (u \circ (v + v_1))_d = (u \circ v)_d + (u \circ v_1)_d ;$$

$$\text{(37)} \qquad ((u + u_1) \circ v)_g = (u \circ v)_g + (u_1 \circ v)_g ;$$

$$\text{(38)} \qquad (u \circ (v + v_1))_g = (u \circ v)_g + (u \circ v_1)_g .$$

Let 0 be the element of $\left(\bigotimes_{j=1}^{m} \mathrm{E}_j\right) \otimes_{\mathrm{A}_m} \left(\bigotimes_{j=1}^{m} \mathrm{E}_{m-j+1}\right)$ a component of which is equal to 0, i.e. $\left(\bigotimes_{j=1}^{m} a_j\right) \otimes \left(\bigotimes_{j=1}^{m} b_{m-j+1}^{*}\right) = 0$ if one of the a_j (resp. b_{m-j+1}^{*}) is equal to 0; under these conditions we have:

$$\text{(39)} \qquad u + 0 = 0 + u = u$$

for any

$$u \in \left(\bigotimes_{j=1}^{m} \mathrm{E}_j\right) \otimes_{\mathrm{A}_m} \left(\bigotimes_{j=1}^{m} \mathrm{E}_{m-j+1}^{*}\right).$$

The identities (27), (32), (33), (34), (35), (36), (37), (38) and (39) show that *the* **Z**-*module* $\left(\bigotimes_{j=1}^{m} \mathrm{E}_j\right) \otimes_{\mathrm{A}_m} \left(\bigotimes_{j=1}^{m} \mathrm{E}_{m-j+1}^{*}\right)$ *provided with a second internal law of composition, defined by the right* (resp. left) *product of composition, is a right* (resp. left) *non commutative ring of composition without unit element for the composition operation.* In particular, *if all rings* $(\mathrm{A}_j)_{1 \leq j \leq m}$ *are equal to the same commutative ring* (resp. field) A, *then the right and left product of composition coincide* (cf. Remark 1, n. 3) *and the right and left rings of composition as defined above coincide with a unique non commutative algebra of composition over* A, *without unit element.*

In our next papers, we shall study the product of composition such as defined above, in the case of topological bimodules, i.e., of bimodules provided with topologies compatible with their structures of bimodules. We shall prove that the topological structures as defined allow to define some topologies for the tensor products of families of topological bimodules.

We also shall study the algebraic and topological properties of the composition of tensor products of finite families of topological bimodules.

Moreover, some of our papers will be devoted to the study of the product of composition of tensor products of finite family of locally convex spaces and to the study of different applications of these products of composition. A summary of the present paper was published in the "Comptes rendus de l'Académie des Sciences de Paris" ([2]).

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