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A mathematical model for physical theories. Nota I

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Fisica matematica. — *A mathematical model for physical theories.* Nota I di ENZO TONTI (*), presentata (**) dal Socio B. FINZI.

RIASSUNTO. — Si presenta un modello matematico per le teorie fisiche basato sulla considerazione di coppie di spazi funzionali messi in dualità da funzionali bilineari e di corrispondenze tra questi spazi. Ognuno di tali spazi funzionali è relativo ad una variabile fisica e le corrispondenze rappresentano le equazioni che legano tra loro le diverse variabili. Dalla struttura degli operatori che descrivono tali corrispondenze si deducono, sotto forma di teoremi, le principali proprietà matematiche del modello.

1.1. INTRODUCTION

Many physical theories exhibit a common mathematical structure that is independent of the physical contents of the theory and is common to discrete and continuum theories, be they of classic, relativistic or quantum nature [1]. The starting point of this structure is the possibility of decomposing the *fundamental equation* [2] of many physical theories in three equations, known in classical fields of the macrocosm as *definition*, *balance* and *constitutive* equations, whose operators enjoy peculiar properties. The properties are as follows: the operator of balance equation is the adjoint, with respect to an opportune bilinear functional, of the operator of definition equation (if the last is linear) or of its Gateaux derivative (if it is nonlinear). Moreover, the operator of constitutive equation is symmetric (when it is linear) or has symmetric Gateaux derivative (when is nonlinear). Such a peculiar decomposition permits us to obtain a profound introspection into the mathematical structure of a theory. The fact that this decomposition can be achieved in a large number of physical theories and the fact that when it exists we can deduce easily a large number of mathematical properties, suggest constructing a mathematical model for physical theories.

1.2. THE MATHEMATICAL MODEL: THE ASSUMPTIONS

Let us suppose we have:

1) a first set of n functions of space and time coordinates $\varphi_k(t, x^1, x^2, x^3)$ (with $k = 1, 2, \cdots, n$), that will be called *configuration variables*. They

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(1) We refer the reader to the paper *On the mathematical structure of large a class of physical theories*, « Rend. Acc. Lincei », *52*, 48–56, denoted by [1].

(2) With this name we indicate the field equation in field theories, the equation of motion in mechanical theories, i.e. the equation relating the configuration of the system with the sources.

can depend only on time variable (3) or on space variables (4) can be of finite or infinite number. They can be real or complex functions, can be the components of a vector, a tensor, a spinor or may do not have special transformation properties. Every given set of these n functions, will be denoted with φ. Any linear function space of elements φ will be denoted with Φ and called *functional configuration space*.

2) a second set of n functions $\sigma_k(t, x^1, x^2, x^3)$ (with $k = 1, 2, \cdots, n$) that will be called *source variables*. They depend from space and time coordinates (5) have the same tensorial order, the same tensorial symmetry properties and the same real or complex nature of configuration variables. Every given set of these variables will be denoted with σ. Any linear function space of elements σ, denoted by Σ, will be called *functional source space*.

3) a bilinear functional defined on the elements of the two function spaces Φ and Σ that will be denoted $\langle \sigma, \varphi \rangle$. It must be such that for every $\sigma \in \Sigma$, different from the null element ϑ, there exists at least one φ such that $\langle \sigma, \varphi \rangle \neq 0$ and analogous requirement on φ. Under these conditions the two spaces are said to be ***put in duality*** by the bilinear functional [2, p. 88].

4) a ***topology*** on the spaces Φ and Σ that makes continuous every linear functional $\langle \sigma_0, \varphi \rangle$ with $\sigma_0 \in \Sigma$ and $\langle \sigma, \varphi_0 \rangle$ with $\varphi_0 \in \Phi$. It can be shown that for every linear functional $l[\varphi]$ continuous with that topology a unique element $\sigma_l \in \Sigma$ can be found so that $l[\varphi] = \langle \sigma_l, \varphi \rangle$ [2, p. 91].

5) a third set of m functions $u_h(t, x^1, x^2, x^3)$ (with $h = 1, 2, \cdots, m$) of space and time coordinates, with $m \geq n$ that we shall call *first kind variables*. Every particular set of such functions will be considered as an element u. Any linear function space formed by elements u will be denoted by U.

6) a fourth set of m functions $v_h(t, x^1, x^2, x^3)$ (with $h = 1, 2, \cdots, m$) of space and time coordinates that we shall call *second kind variables* such that every v_h has the same tensor nature and the same tensorial symmetries of the first kind variables u_h. Every particular set of such functions will be denoted by u. Any linear function space formed by elements v will be denoted by V.

7) a *bilinear functional* defined on the elements of the two spaces U and V denoted by $\langle v, u \rangle$ that satisfies the same requirements of point 3).

8) a ***topology*** for U and V spaces with the same requirements of point 4).

(3) As the lagrangian coordinates in mechanics and the extensive parameters in the irreversible thermodynamics of discrete systems.

(4) As in time-independent field theories (static and stationary fields).

(5) Source variables can depend on space and time coordinates either directly as when they are assigned (fixed or impressed sources) or indirectly as when they are linked with configuration variables of other systems (interaction) or with those of the same system (self-interaction).

Up to this point we have two pairs of function spaces in duality equipped with suitable topologies. The need to introduce a topology arises from the fact that we wish to treat subjects as stability, perturbations, convergence of iterative methods, error bounds in approximate methods and existence of solution. About mappings among these spaces we suppose to have:

9) a mapping D, generally nonlinear, between some subset $\mathfrak{D}(D) \subseteq \Phi$ (its domain) of the functional configuration space and a subset $\mathfrak{R}(D) \subseteq U$ (its range) of the function space U of first kind variables. When $m > n$ the operator D is a gradient-like operator. The equation $u = D\varphi$ will be called *definition equation*;

10) a mapping C, generally nonlinear, between a subset $\mathfrak{D}(C) \supseteq \mathfrak{R}(D)$ of the U–space and a subset $\mathfrak{R}(C)$ of the V–space. The operator C will be supposed *symmetric*, if linear, i.e. $\langle Cu', u'' \rangle = \langle Cu'', u' \rangle$ or with *symmetric Gateaux derivative*, if nonlinear, i.e. $\langle C'_u u', u'' \rangle = \langle C'_u u'', u' \rangle$. Moreover it is supposed that C does not contain the configuration variable φ. The equation $v = Cu$ will be called *constitutive equation.*

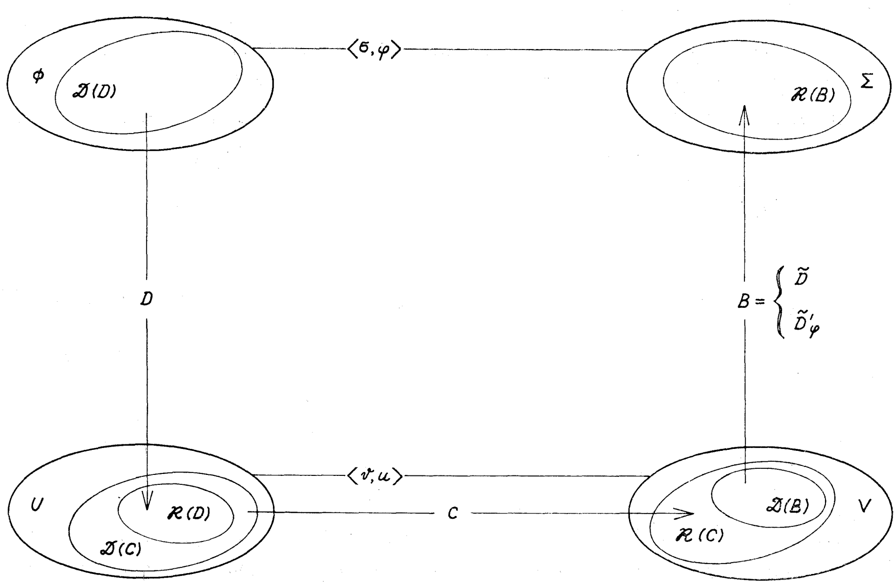

Fig. 1.

11) a *linear* mapping B between some subset $\mathfrak{D}(B) \subseteq \mathfrak{R}(C)$ of the V–space and a subset $\mathfrak{R}(B)$ of the Σ–space that be the *adjoint* of the mapping D (if D is linear) or be the adjoint of its linear Gateaux derivative (if D is nonlinear) (6), we shall use the notations $B = \tilde{D}$ and $B = \tilde{D}'_\varphi$ respectively.

(6) In physical theories the operator D is generally not continuous, being often a differential operator working on a Banach space (in particular on Hilbert and Sobolev spaces). It follows that the Gateaux derivative is not continuous in this case. Some Authors speak of Gateaux derivative only when continuity is assured [7, p. 40] [8, p. 114]. This usage is very restrictive: we adhere to the more general definition (see for ex. Tapia in [8, p. 51]).

When $m > n$ the operator B is a divergence-like operator. The equation $Bv = \sigma$ will be called *balance equation*.

We emphasize the fact that of the three mappings we shall take as primitive, only two of them are independent i.e. D and C. In the sequel will be shown that the mathematical properties of the model rest upon the properties of these two operators. The scheme of fig. 1 summarizes what we have said up to now.

1.3. The mathematical model: first properties. *a*) Fundamental equation

The sequence of mappings $D : \Phi \mapsto U$, $C : U \mapsto V$, $B : V \mapsto \Sigma$ induce a mapping $F = BCD : \Phi \mapsto \Sigma$ we shall call *fundamental mapping*. The corresponding fundamental equation has the form

$$\tilde{D}CD\varphi = \sigma \qquad \tilde{D}'_{\varphi}\, CD\varphi = \sigma \tag{1.3.1}$$

in the linear and nonlinear case respectively. The fundamental mapping F enjoys many properties: we shall consider in this paper the case in which D and C are *linear* operators. In the second part we shall deal with the nonlinear case.

Theorem 1: *If* D *and* C *are linear operators the operator* F *is symmetric.*

Proof:

$$\langle F\varphi', \varphi''\rangle = \langle \tilde{D}CD\varphi', \varphi''\rangle = \langle CD\varphi', \tilde{\tilde{D}}\varphi''\rangle = \langle C\tilde{\tilde{D}}\varphi'', D\varphi'\rangle = \langle \tilde{D}C\tilde{\tilde{D}}\varphi'', \varphi'\rangle . \tag{1.3.2}$$

But $\tilde{\tilde{D}} \supseteq D$ [9, p. 168] and then if $\varphi'' \in \mathfrak{D}(F)$

$$\langle F\varphi', \varphi''\rangle = \langle \tilde{D}CD\varphi'', \varphi'\rangle = \langle F\varphi'', \varphi'\rangle . \tag{1.3.3}$$

From the symmetry of F follow two properties: they are

Theorem 2 (variational formulation): *if the operator* F *is symmetric and* σ *does not depend on* φ, *the solutions of the fundamental equation, when it exist, make stationary the functional*

$$S[\varphi] \overset{\text{def}}{=} \frac{1}{2} \langle CD\varphi , D\varphi\rangle - \langle \sigma , \varphi\rangle . \tag{1.3.4}$$

Proof:

$$\delta S[\varphi] = \langle CD\varphi , D\delta\varphi\rangle - \langle \sigma , \delta\varphi\rangle = \langle \tilde{D}CD\varphi - \sigma , \delta\varphi\rangle = 0 . \tag{1.3.5}$$

This theorem, stated in other words, asserts that the fundamental equation is the Euler–Lagrange equation of an action functional. We thus see that the existence of an action functional for the fundamental equation, that is assumed as postulate in field theory, is here deduced as theorem.

THEOREM 3 (RECIPROCITY THEOREM): *if the operator* F *is symmetric let us be* σ' *and* σ'' *two different sources and* φ', φ'' *two corresponding solutions then*

$$\langle \sigma', \varphi'' \rangle = \langle \sigma'', \varphi' \rangle. \tag{1.3.6}$$

Proof:

$$\langle \sigma', \varphi'' \rangle = \langle F\varphi', \varphi'' \rangle = \langle F\varphi'', \varphi' \rangle = \langle \sigma'', \varphi' \rangle. \qquad \text{(q.e.d.)} \tag{1.3.7}$$

A frequent case is that the operator C be definite positive. When this happens we have the following properties:

THEOREM 4 (MINIMUM OF THE FUNCTIONAL): *if* C *is a positive definite operator, i.e.* $\langle Cu , u \rangle > 0$ *for* $u \neq \theta$ *then the solution of the fundamental equation, when exists, makes minimum the action functional* S *of Theorem 2.*

Proof: being $\delta S[\varphi] = \langle \tilde{D}CD\varphi - \sigma , \delta\varphi \rangle$ will be

$$\delta^2 S[\varphi] = \langle \tilde{D}CD\delta\varphi , \delta\varphi \rangle = \langle CD\delta\varphi , D\delta\varphi \rangle = \langle C\delta u , \delta u \rangle > 0 . \tag{1.3.8}$$

THEOREM 5: *if* C *is a positive definite operator the fundamental operator has the same null manifold of the definition operator*:

$$\mathfrak{N}(F) = \mathfrak{N}(D)$$

Proof:

$$\varphi_0 \in \mathfrak{N}(F) \Rightarrow F\varphi_0 = 0 \Rightarrow \langle F\varphi_0 , \varphi_0 \rangle = \langle CD\varphi_0 , D\varphi_0 \rangle = 0 \Rightarrow \Rightarrow D\varphi_0 = 0 \Rightarrow \varphi_0 = \mathfrak{N}(D) \tag{1.3.9}$$

$$\varphi_0 \in \mathfrak{N}(D) \Rightarrow D\varphi_0 = 0 \Rightarrow CD\varphi_0 = 0 \Rightarrow \tilde{D}CD\varphi_0 = 0 \Rightarrow \Rightarrow F\varphi_0 = 0 \Rightarrow \varphi_0 = \mathfrak{N}(F) . \tag{1.3.10}$$

From this theorem follows as a lemma the

THEOREM 6: (UNIQUENESS). *If the operator* C *is positive definite and the operator* D *has no null manifold, the solution of the fundamental equation, when exists, is unique.*

The existence of a null manifold of the definition operator D implies the existence of a compatibility condition on the source term σ irrespectively of the positive definite character of the operator C.

THEOREM 7: *if the definition operator* D *has a null manifold, denoted with* $\varphi_0 = L\chi$ *the general solution of the homogeneous equation* $D\varphi = 0$ *then in order that the fundamental problem admits a solution must be* $\tilde{L}\sigma = 0$ (7).

Proof:

$$DL\chi \equiv 0 \Rightarrow \tilde{L}\tilde{D} = 0 \Rightarrow \tilde{L}\tilde{D}CD\varphi = \tilde{L}\sigma = 0 . \tag{1.3.11}$$

(7) Because the symmetry of C does not enter in this theorem while it is essential for the variational formulation (see Theorem 2) we see that the link between gauge invariance and conservation laws is essentially of non-variational nature. Neither theorem requiring a variational principle is then very demanding.

The property $DL\chi = 0$ is commonly known in physics as gauge invariance. The theorem establishes a link between the gauge invariance of first kind variables and the existence of compatibility conditions that usually mean conservation laws [3].

1.4 The mathematical model: *b*) Canonical form

If the constitutive mapping is one to one we can consider the inverse mapping C^{-1}. In this case we can reduce the three basic equations to the following two equations

$$D\varphi = C^{-1} v \quad , \quad \tilde{D} v = \sigma . \tag{1.4.1}$$

These two sets will be called the *canonical system.*

Theorem 8 (variational formulation): *the solutions of the canonical system, with σ assigned, make stationary the functional*

$$\bar{S}[\varphi , v] \overset{\text{def}}{=} \langle v , D\varphi \rangle - \frac{1}{2} \langle v , C^{-1} v \rangle - \langle \sigma , \varphi \rangle \tag{1.4.2}$$

Proof:

$$\delta \bar{S}[\varphi , v] = \langle \delta v , D\varphi - C^{-1} v \rangle + \langle \tilde{D} v - \sigma , \delta\varphi \rangle = 0 . \tag{1.4.3}$$

The functional $\bar{S}[\varphi , v]$ will be called the *canonical action functional.*

The canonical equations can be written in a matrix–differential form as follows

$$\left[\begin{array}{c|c} 0 & \tilde{D} \\ \hline D & -C^{-1} \end{array}\right] \left[\begin{array}{c} \varphi \\ \hline v \end{array}\right] = \left[\begin{array}{c} \sigma \\ \hline 0 \end{array}\right] . \tag{1.4.4}$$

If we introduce two vectors $\psi = (\varphi_1 , \cdots , \varphi_n ; v_1 , \cdots , v_m)$ and $\chi = (\sigma_1 , \cdots , \sigma_n ; 0 , \cdots , 0)$ putting

$$L = \left[\begin{array}{c|c} 0 & \tilde{D} \\ \hline D & 0 \end{array}\right] \qquad K = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -C^{-1} \end{array}\right] \tag{1.4.5}$$

the canonical system can be written as

$$L\psi + K\psi = \chi . \tag{1.4.6}$$

Often D is a first order linear operator: in this case the matrix–differential operator L can be decomposed in the form

$$L = \sum_{0}^{3}{}_{\alpha} L_{\alpha} \frac{\partial}{\partial x^{\alpha}} \tag{1.4.7}$$

where L_{α} denotes some square matrices of $(m + n)^2$ elements. The canonical system then assumes the typical form

$$\sum_{0}^{3}{}_{\alpha} L_{\alpha} \frac{\partial}{\partial x^{\alpha}} \psi + K\psi = \chi \tag{1.4.8}$$

used in the matrix–algebraic approach to the relativistic theory of particles of arbitrary spin [4, p. 378] [5, p. 270] [6, p. 143].

THEOREM 9 (SYMMETRY OF THE OPERATORS L AND K). *The matrix–differential operator* L *and the operator* K *are symmetric with respect to the bilinear functional*

$$\langle \chi , \psi \rangle \overset{\text{def}}{=} \langle \sigma , \varphi \rangle + \langle v , u \rangle . \tag{1.4.9}$$

Proof:

$$\langle \mathrm{L}\psi', \psi'' \rangle = \langle \tilde{\mathrm{D}} v', \varphi'' \rangle + \langle v'', \mathrm{D}\varphi' \rangle = \langle v', \mathrm{D}\varphi'' \rangle + \langle \tilde{\mathrm{D}} v'', \varphi' \rangle = \langle \mathrm{L}\psi'', \psi' \rangle \tag{1.4.10}$$

$$\langle \mathrm{K}\psi' , \psi'' \rangle = \langle -\mathrm{C}^{-1} v' , v'' \rangle = \langle -\mathrm{C}^{-1} v'' , v' \rangle = \langle \mathrm{K}\psi'' , \psi' \rangle \tag{1.4.11}$$

Using the bilinear functional (1.4.9) the canonical action functional (1.4.2) can be written

$$\bar{\mathrm{S}} [\psi] = \frac{1}{2} \langle \mathrm{L}\psi , \psi \rangle + \frac{1}{2} \langle \mathrm{K}\psi , \psi \rangle - \langle \chi . \psi \rangle \tag{1.4.12}$$

(the proof is straightforward).

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