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Topologia. — *An axiomatic topological characterization of Hilbert space.* Nota di JOHAN SWART, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Recentemente J. de Groot [3] ha dato una sufficiente caratterizzazione topologica assiomatica degli spazi I^n [cubo metrico n dimensionale], R^n [spazio euclideo n dimensionale], I^∞ [cubo di Hilbert], S^n [sfera n dimensionale] e P^n [spazio proiettivo n dimensionale].

Lo scopo di questa Nota è di dare una caratterizzazione topologica assiomatica di R^∞ [prodotto di un'infinità numerabile di rette reali] basata su una veduta e sul metodo di J. de Groot.

Gli assiomi di J. de Groot per I^∞ sono modificati per il caso non compatto.

In vista del noto risultato di R. D. Anderson che lo spazio di Hilbert è omeomorfo col prodotto di un'infinità numerabile di linee rette si ottiene la caratterizzazione topologica dello spazio di Hilbert [Vedere R. D. Anderson [1], e R. D. Anderson–R. H. Bing [2]].

Two basic concepts introduced by J. de Groot are the following:

A topological space X is said to be *2-compact* if there exists an open subspace for X such that every subbasic open cover of X has a subcover by at most two elements. (In this connection we would like to mention a conjecture of J. de Groot, proved by J. L. O'Connor [5] which states that: "every compact metrizable space is 2-compact").

Secondly, a family $\mathfrak{F}$ of open subsets of X is called *comparable* if for all $S_0, S_1, S_2 \in \mathfrak{F}$ the following property holds:

$$\left.\begin{array}{l} S_0 \cup S_1 = X \\ S_0 \cup S_2 = X \end{array}\right\} \Rightarrow S_1 \subsetneqq S_2$$

where "$S_1 \subsetneqq S_2$" stands for "$S_1 \subset S_2$ or $S_1 \supset S_2$".

A basic property of $I^\infty(I^n)$ is that $I^\infty(I^n)$ is 2-compact relative to a comparable open subbase. On the other hand $R^\infty(R^n)$ is not 2-compact. However, the open subbase $\mathfrak{S}$ of $R^\infty(R^n)$ consisting of all sets of the form $p_i^{-1}(-\infty, a)$ and $p_i^{-1}(a, \infty)$, $a \in R$ and all i has the property that every open cover of the space which contains at least one member of the form $p_i^{-1}(-\infty, a)$ and one of the form $p_i^{-1}(a, \infty)$ for each i has a subcover by two of these open sets.

Clearly $\mathfrak{S}$ is also comparable.

The idea of looking at such covers was inspired by the concept of "pairwise open covers" as used by Fletcher, Hoyle and Patty [4] in their definition of "pairwise compactness" for bitopological spaces. The above observation enables us to use J. de Groot's proof as a model for ours.

(*) Nella seduta del 12 febbraio 1972.

The Author is sincerely grateful to prof. J. de Groot and dr. A. Verbeek for having made available to him the, as yet unpublished, results referred to above. The paper was written while the Author was a research student of dr. R. J. Wille. The problem of characterizing Hilbert space was suggested by dr. R. J. Wille whose many helpful suggestions and encouragement have made the writing of this paper possible.

In proving the Theorem below we work with closed subbases. By a *linked* family $\mathfrak{F}$ of subsets of X we shall mean a family with the property that the intersection of any pair of members is non-empty.

THEOREM. *A topological space* X *is homeomorphic to the countable product of real lines if and only if* X *satisfies the following conditions*:

(1) X *is* T_1;

(2) X *is connected*;

(3) X *has a countable closed subbase* $\mathfrak{S}$ *which does not contain* X *nor* Φ *and which satisfies* (4), (5) *and* (6);

(4) $\mathfrak{S}$ *is comparable, i.e.* $\forall S, S_1, S_2 \in \mathfrak{S}$,

$$\left.\begin{matrix} S \cap S_1 = \Phi \\ S \cap S_2 = \Phi \end{matrix}\right\} \Rightarrow S_1 \subseteqq S_2 \text{ or } S_1 \supseteqq S_2;$$

Let $\mathfrak{E}$ *denote the family of all maximal linearly ordered* (*by inclusion*) *subsets* E *of* $\mathfrak{S}$.

(5) $\forall E \in \mathfrak{E}$, $\cap E = \Phi$ *and* $\cup E = X$;

(6) *All linked* $\mathfrak{F} \subset \mathfrak{S}$ *which satisfy* $\mathfrak{F} \cap E \neq \Phi$ *and* $E \not\subset \mathfrak{F}$ $\forall E \in \mathfrak{E}$ *have the property that* $\cap \mathfrak{F} \neq \Phi$.

Proof. It is clear that $\prod_{i=1}^{n} R_i$ and $\prod_{i=1}^{\infty} R_i$ satisfy the above conditions—a suitable closed subbase consists of all $p_i^{-1}(-\infty, r]$ and $p_i^{-1}[r, \infty)$ for all i, where r is a rational number.

Suppose then that X is T_1, connected and has a closed subbase $\mathfrak{S}$ which is countable, does not contain X nor Φ, is comparable and satisfies (5) and (6). The proof is split up into various Lemmas, starting with an important separation property and following the proof of J. de Groot.

LEMMA 1. $\forall S \in \mathfrak{S}$, $\forall x \notin S$ $\exists S' \in \mathfrak{S}$ *such that* $x \in S'$ *and* $S \cap S' = \Phi$.

Proof. Let $S \in \mathfrak{S}$, $x \in X - S$ and consider $\tau = \{T \mid x \in T \in \mathfrak{S}\}$. Since X is T_1, $\tau \neq \Phi$ and in fact $\cap \tau = \{x\}$. Also $\tau \cup \{S\}$ contains a member of each $E \in \mathfrak{E}$ (by (5)) and no E is contained in $\tau \cup \{S\}$. Now $E \subset \tau$ contradicts $\cap E = \Phi$. Suppose next that $S \in E^*$ and that all members of E^* other than S are contained in τ. By linear order in E^* this would imply that S is contained in every other member of E^* and hence contradict $\cap E^* = \Phi$.

Thus $\forall E \in \mathfrak{E}$, $E \not\subset \tau \cup \{S\}$. If $\tau \cup \{S\}$ is linked it follows by (6) that $\cap (\tau \cup \{S\}) \neq \Phi$; hence $x \in S$, a contradiction. Hence $\tau \cup \{S\}$ is not linked, i.e. there exists $S' \in \tau$ with $x \in S'$ and such that $S \cap S' = \Phi$.

LEMMA 2. $\forall S, S' \in \mathcal{S}$, $S \cap S' = \Phi \Rightarrow \exists T, T' \in \mathcal{S}$ *such that* $T' \cap S = \Phi = T \cap S'$, $T \cup T' = X$, $S \subsetneqq T$ *and* $S' \subsetneqq T'$.

Proof. Since X is connected $\exists x \in X - S \cup S'$.

By Lemma 1 $\exists T', T \in \mathcal{S}$ such that $x \in T' \cap T$ and $S \cap T' = \Phi = S' \cap T$.

By the comparability of $\mathcal{S}$,

$$\left.\begin{array}{l} S \cap S' = \Phi \\ T \cap S' = \Phi \end{array}\right\} \Rightarrow S \subsetneqq\!\!\!\supset T .$$

However $x \in T$, $x \notin S$ and so $S \subsetneqq T$.

Similarly $S' \subsetneqq T'$.

Furthermore $T \cup T' = X$:

Suppose not, then $\exists y \in X - T \cup T'$ and by Lemma 1 $\exists U' \in \mathcal{S}$ such that $y \in U'$ and $T \cap U' = \Phi$.

By the comparability of $\mathcal{S}$,

$$\left.\begin{array}{l} S \cap U' = \Phi \\ S \cap T' = \Phi \end{array}\right\} \Rightarrow T' \subsetneqq\!\!\!\supset U' .$$

However, $x \in T$ and $T \cap U' = \Phi$ so that $x \in T' - U'$ and $y \in U' - T'$. Thus $T' \not\subsetneqq\!\!\!\supset U'$, a contradiction, and hence $T \cup T' = X$.

We prove next that the sets E in $\mathcal{E}$ form a partition of $\mathcal{S}$.

LEMMA 3. *The comparability relation "$\subsetneqq\!\!\!\supset$" on $\mathcal{S}$ is an equivalence relation, and the equivalence classes are precisely the maximal linearly ordered* E's.

Proof. It is clear that "$\subsetneqq\!\!\!\supset$" is reflexive and symmetric. We show that $S \subsetneqq\!\!\!\supset T$ and $T \subsetneqq\!\!\!\supset U \Rightarrow S \subsetneqq\!\!\!\supset U$.

(i) $S \subset T$, $T \subset U \Rightarrow S \subset U$ and $S \supset T$, $T \supset U \Rightarrow S \supset U$.

(ii) Let $S \subset T$, $T \supset U$. Since $X \notin \mathcal{S}$ $\exists x \in X - T$ and by Lemma 1 $\exists T' \in \mathcal{S}$ such that $x \in T'$ and $T \cap T' = \Phi$.

By the comparability of $\mathcal{S}$,

$$\left.\begin{array}{l} T' \cap S = \Phi \\ T' \cap U = \Phi \end{array}\right\} \Rightarrow S \subsetneqq\!\!\!\supset U .$$

(iii) Let $S \supset T$, $T \subset U$ and suppose $S \not\subsetneqq\!\!\!\supset T$. Choose $x \in S - U$ and $y \in U - S$. Apply Lemma 1 to obtain $U', S' \in \mathcal{S}$ such that $x \in U'$, $y \in S'$, $U \cap U' = \Phi = S \cap S'$.

By comparability of $\mathcal{S}$,

$$\left.\begin{array}{l} T \cap S' = \Phi \\ T \cap U' = \Phi \end{array}\right\} \Rightarrow S' \subsetneqq\!\!\!\supset U' .$$

However, $x \in U' - S'$ and $y \in S' - U'$, a contradiction; hence $S \subsetneqq\!\!\!\supset T$.

Furthermore, since each equivalence class is linearly ordered by inclusion and consists of all members of $\mathfrak{S}$ which are equivalent under the relation "$\subsetneqq$", it is maximal and hence the set of equivalence classes is precisely the set $\mathfrak{E}$ of all maximal linearly ordered subsets of $\mathfrak{S}$.

LEMMA 4. *For each* $E \in \mathfrak{E}$ *there exists a unique* $E' \in \mathfrak{E}$ *which satisfies* $\forall S \in E \ \exists S' \in E'$ *such that* $S \cap S' = \Phi$.

Furthermore $(E')' = E$ and hence the equivalence classes are paired off.

Proof. Consider $E^* = \{S' \in \mathfrak{S} \mid \exists S \in E \text{ such that } S \cap S' = \Phi\}$. Let $S_1', S_2' \in E^*$; then there exist $S_1, S_2 \in E$ such that $S_1 \cap S_1' = \Phi = S_2 \cap S_2'$.

Now $S_1 \subsetneqq S_2$, suppose $S_1 \subset S_2$.

By the comparability of $\mathfrak{S}$,

$$\left.\begin{array}{l} S_1 \cap S_1' = \Phi \\ S_1 \cap S_2' = \Phi \end{array}\right\} \Rightarrow S_1' \subsetneqq S_2'.$$

Hence $E^* \subset E'$ for some $E' \in \mathfrak{E}$ and in fact $E^* = E'$;

Let $T' \in E'$. Then $\exists S' \in E^*$ such that $T' \supset S'$.

Since $X \notin \mathfrak{S}$ $\exists x \notin T'$ and by Lemma 1 $\exists T \in \mathfrak{S}$ such that

$$x \in T \quad \text{and} \quad T \cap T' = \Phi.$$

Since $S' \in E^*$ $\exists S \in E$ such that $S \cap S' = \Phi$.

By comparability,

$$\left.\begin{array}{l} S \cap S' = \Phi \\ T \cap S' = \Phi \end{array}\right\} \Rightarrow S \subsetneqq T.$$

Since E is an equivalence class, $T \in E$ and by definition of E^*, $T' \in E^*$ so that also $E' \subset E^*$. Thus $E^* = E'$.

It is clear that E' is constructed in a unique way.

Let $S \in E$ and choose $x \in X - S$. By Lemma 1 $\exists S' \in \mathfrak{S}$ such that $S \cap S' = \Phi$ and since $E = E^*$ is unique, $S' \in E'$.

Finally, $(E')' = \{T \in \mathfrak{S} \mid \exists S' \in E' \text{ such that } T \cap S' = \Phi\}$.

Now there exist $S \in E$, $S' \in E'$ such that $S \cap S' = \Phi$ and hence $S \in (E')'$. However, the equivalence classes are disjoint so that $E = (E')'$.

LEMMA 5. $\forall S \in E \ \forall S' \in E'$, $S \cap S' = \Phi$ *or* $S \cup S' = X$.

Proof. Let $S \in E$, $S' \in E'$ and suppose $S \cap S' \neq \Phi$ and $S \cup S' \neq X$. Choose $x \in S \cap S'$ and $y \in X - S \cup S'$. By Lemma 1 and the uniqueness of E' $\exists T' \in E'$ such that $x \in T'$ and $S \cap T' = \Phi$. Since $T', S' \in E'$, $T' \subsetneqq S'$. However, $x \in S' - T'$ and $y \in T' - S'$, a contradiction. Hence result.

LEMMA 6. *Each* $E \in \mathfrak{E}$ *has the following order properties*:

(i) E *has no 'largest'*;

(ii) E *has no 'smallest'*;

(iii) $\forall S_0, S_1 \in E$ *with* $S_0 \subsetneq S_1$ *the followig statements are true*:

(a) $\exists S_0' \in E'$ *such that* $S_0 \cap S_0' = \Phi$ *and* $S_1 \cup S_0' = X$;

(b) $\exists T_0 \in E$ *such that* $S_0 \subsetneq T_0 \subsetneq S_1$;

(c) $S_0 \subsetneq \mathit{int}\, S_1$.

Proof. (i) Let $S \in E$. By Lemma 1 and the uniqueness of E' $\exists S' \in E'$ such that $S \cap S' = \Phi$. By Lemma 2 and uniqueness of E $\exists T \in E$ such that $S \subsetneqq T$, and hence E has no largest.

(ii) Follows by condition (5) since the existence of a smallest member would contradict $\cap E = \Phi$.

(iii) Let $S_0, S_1 \in E$ with $S_0 \subsetneqq S_1$ and choose $x \in S_1 - S_0$.

By Lemma 1 $\exists S'_0 \in \mathcal{S}$ such that $x \in S'_0$ and $S_0 \cap S'_0 = \Phi$. Clearly $S'_0 \in E'$ and since $x \in S'_0 \cap S_1$ it follows by Lemma 5 that $S'_0 \cup S_1 = X$. Also $X - S'_0 \subsetneqq S_1$.

By Lemma 2 $\exists T_0 \in \mathcal{S}$ such that $T_0 \cap S'_0 = \Phi$ and $S_0 \subsetneqq T_0$.

Hence $S_0 \subsetneqq T_0 \subset X - S'_0 \subsetneqq S_1$ and clearly also $T_0 \in E$.

Since $X - S'_0$ is open it follows that $S_0 \subsetneqq \operatorname{int} S_1$.

Since $\mathcal{S}$ is countable the number of equivalence classes is also countable. Let $\{E_n, E'_n\}_{n \in A}$ be an enumeration of the pairs of equivalence classes. We will prove that X can be embedded homeomorphically in $\Pi\{R_n \mid n \in A\}$. Before proving this result we need two more Lemmas.

LEMMA 7. *Let* E *denote an arbitrary* E_n *in the above enumeration of the pairs of equivalence classes. Then there exists a continuous map* $f: X \to R$ *with the property that* $E = \{f^{-1}(-\infty, r\} \mid r \in Q\}$ *where* Q *denotes the set of rational numbers.*

Proof. Let $\{r_i\}_{i \in N}$ be a listing of the rational numbers with $r_1 < r_2$ and let $\{S_j\}_{j \in N}$ be a listing of the members of E with $S_1 \subset S_2$.

Throughout the proof of Lemma 7, "$A \subset B$" shall mean "$A \subsetneqq B$".

We rearrange the S_j's in a family $\{S_{r_n}\}_{n \in N}$ such that

$$r_k, r_n \in Q, \quad r_k < r_n \Rightarrow S_{r_k} \subset \operatorname{int} S_{r_n}.$$

By induction we define S_{r_n}, $n \geqq 3$ as follows:

(i) If $r_n < r_k$ $\forall k < n$, put $p_n = \min\{r_k \mid \varkappa < n\}$ and let S_{r_n} be the first in the enumeration of $E = \{S_j\}_{j \in N}$ such that $S_{r_n} \subset S_{p_n}$ (Lemma 6 (ii)).

(ii) If $r_n > r_k$ $\forall k < n$, put $q_n = \max\{r_k \mid \varkappa < n\}$ and let S_{r_n} be the first in the enumeration $\{S_j\}_{j \in N}$ such that $S_{r_n} \supset S_{q_n}$ (Lemma 6 (i)).

(iii) If neither (i) nor (ii) applies then take the largest r_i and smallest r_j with $i, j < n$ and which satisfy $r_i < r_n < r_j$. Let S_{r_n} be the first in the enumeration $\{S_j\}_{j \in N}$ such that $S_{r_i} \subset S_{r_n} \subset S_{r_j}$ (Lemma 6 (iii)).

Define $f: X \to R$ by $f(x) = \inf\{r_n \mid x \in S_{r_n}\}$.

Since each $x \in X$ belongs to some S_{r_n} but not to all of them, the function f is well defined (c.f. condition (5)).

We show that $f^{-1}(-\infty, r_n]$ and $f^{-1}[r_n, \infty)$ are closed in X for $r_n \in Q$.

(a) $f^{-1}(-\infty, r_n] = S_{r_n}$:

$$x \in S_{r_n} \Rightarrow f(x) = \inf\{r_m \mid x \in S_{r_m}\} \leqq r_n \Rightarrow x \in f^{-1}(-\infty, r_n] .$$

Also $x \in f^{-1}(-\infty, r_n] \Rightarrow f(x) \leqq r_n \Rightarrow x \in S_{r_m} \quad \forall r_m > r_n$.

(If $x \notin S_{r_M}$ for some $r_M > r_n$, then r_M is a lower bound greater than r_n—a contradiction since the infimum $\leqq r_n$).

Suppose $x \notin S_{r_n}$.

By Lemma 1 and uniqueness of E′ $\exists S' \in E'$ such that $x \in S'$ and $S_{r_n} \cap S' = \Phi$.

By Lemma 2 and uniqueness of E $\exists T \in E$ such that $T \cap S' = \Phi$ and $S_{r_n} \subset T$.

Since $T \cap S' = \Phi$, $x \notin T$ which gives a contradiction since $T = S_{r_m}$ for some $r_n > r_n$. Thus $x \in S_{r_n}$ and hence $S_{r_n} = f^{-1}(-\infty, r_n]$.

(b) $f(x) \geqq r_n \Longleftrightarrow x \in X - \text{int } S_{r_m} \quad \forall r_m < r_n$:

$$\begin{aligned} f(x) \geqq r_n &\Rightarrow \inf\{r_m \mid x \in S_{r_m}\} \geqq r_n \\ &\Rightarrow x \notin S_{r_m} \qquad \forall r_m < r_n \\ &\Rightarrow x \notin \text{int } S_{r_m} \qquad \forall r_m < r_n . \end{aligned}$$

Also $x \in X - \text{int } S_{r_m} \quad \forall r_m < r_n$

$$\begin{aligned} &\Rightarrow x \notin \text{int } S_{r_m} \qquad \forall r_m < r_n \\ &\Rightarrow x \notin S_{r_m} \qquad \forall r_m < r_n . \end{aligned}$$

(If $x \in S_{r_k}$ for some $r_k < r_n$, then we can choose $r_l \in Q$ such that $r_k < r_l < r_n$ and hence $x \notin \text{int } S_{r_l}$—this gives a contradiction since $r_k < r_l$ and hence $S_{r_k} \subset \text{int } S_{r_l}$)

$$\Rightarrow f(x) \geqq r_n .$$

Thus $f^{-1}[r_n, \infty) = \cap\{(X - \text{int } S_{r_m}) \mid r_m < r_n\}$ which is closed.

Hence f is continuous since all sets of the form $(-\infty, r_n]$ and $[r_n, \infty)$ with $r_n \in Q$ form a closed subbase for R.

COROLLARIES (i) *$f: X \to R$ is a surjection.*

(ii) *If $E = \{f^{-1}(-\infty, r_n] \mid r_n \in Q\}$ then the following statements are true:*

(a) *Given any $a \in R$ $\exists S' \in E'$ such that $f^{-1}[a, \infty) \subset S'$;*

(b) *Each $S' \in E'$ is of the form $f^{-1}[m, \infty)$, some $m \in R$;*

(c) *If $a, b \in R$ with $a < b$ then $\exists c \in R$ such that $a < c < b$ and $f^{-1}[c, \infty) \in E'$.*

Proofs. (i) $f: X \to R$ is a surjection since X is connected and since for any $r_n \in Q$ $\exists x, y \in X$ such that $f(x) \geqq r_n$ and $f(y) \leqq r_n$. Thus $f(X)$ cannot be an interval, which is bounded below or above and hence $f(X) = R$.

(ii) (a) Let $r_n \in Q$ with $r_n < a$ and let $S = f^{-1}(-\infty, r_n] \in E$. Let $r_n < r_m < a$ for $r_m \in Q$ and let $S_1 = f^{-1}(-\infty, r_m] \in E$.

By Lemma 6 $\exists S' \in E'$ such that $S \cap S' = \Phi$ and $S_1 \cup S' = X$.

Hence $S' \supset f^{-1}(r_m, \infty) \supset f^{-1}[a, \infty)$.

(b) Let $S' \in E'$. There exists $S \in E$ such that $S' \cap S = \Phi$. Let $S = f^{-1}(-\infty, r]$.

Suppose $\exists a, b \in R$ with $r < a < b$ and such that $f^{-1}\{a\} \subset S'$ and $f^{-1}\{b\} \not\subset S'$.

Choose $r_0 \in Q$, $a < r_0 < b$ and let $S_0 = f^{-1}(-\infty, r_0] \in E$.

Since $S_0 \cap S' \neq \Phi$ it follows by Lemma 5 that $S_0 \cup S' = X$.

However, $f^{-1}\{b\} \not\subset S_0 \cup S'$.

Thus if $f^{-1}\{a\} \subset S'$ and if $b > a$ then $f^{-1}\{b\} \subset S'$.

Let $m = \inf\{f(x) \mid x \in S'\}$ and let $m = f(x_0)$, $x_0 \in X$.

We show that $S' = f^{-1}[m, \infty)$.

Suppose $x_0 \notin S'$. By Lemma 1 and uniqueness of E $\exists S_1 \in E$ such that $x_0 \in S_1$ and $S_1 \cap S' = \Phi$.

Clearly $S_1 = f^{-1}(-\infty, k]$ for some $k \geqq m$.

By Lemma 2 and uniqueness of E $\exists T \in E$ such that $T \cap S' = \Phi$ and $S_1 \subsetneqq T$.

Thus $T = f^{-1}(-\infty, q]$ where $m \leqq k < q$ and since $T \cap S' = \Phi$, $q < f(x)$ $\forall x \in S'$.

However, $m < q$ contradicts the fact that m is the greatest lower bound of the set $\{f(x) \mid x \in S'\}$.

Thus $x_0 \in S'$ and hence $S' = f^{-1}[m, \infty)$.

(c) Let $p, q \in Q$ with $a < p < q < b$ and let $S_0 = f^{-1}(-\infty, p] \in E$, and $S_1 = f^{-1}(-\infty, q] \in E$.

By Lemma 6 (iii) $\exists S' \in E'$ such that $S_0 \cap S' = \Phi$ and $S_1 \cup S' = X$.

Thus $S' \supset f^{-1}(q, \infty)$ and hence $S' = f^{-1}[c, \infty)$ for $p < c \leqq q$.

LEMMA 8. *For any two different, non-paired equivalence classes* E_1 *and* E_2 *the following holds*:

$$\forall S_1 \in E_1 \quad \forall S_2 \in E_2 \qquad S_1 \cap S_2 \neq \Phi .$$

Proof. We have $E_1 \neq E_2$ and $E_2 \neq E_1'$.

Let $S_1 \in E_1$ and $S \in \mathcal{S}$ be such that $S_1 \cap S = \Phi$. By the uniqueness of E_1', $S \in E_1'$ and since $E_1' \neq E_2$, $E_1' \cap E_2 = \Phi$ and hence $S \notin E_2$. Hence result.

Proof of the Theorem. Let $\{E_n, E_n'\}_{n \in A}$ be an enumeration of the pairs of equivalence classes. For each $n \in A$ we define in accordance, with Lemma 7, a continuous surjection $f_n : X \to R$.

The diagonal map

$$\Delta = \mathop{\Delta}_{n \in A} f_n : X \to \Pi\{R_n \mid n \in A\}$$

$$x \mapsto (f_n(x))_{n \in A}$$

gives the required homeomorphism:

(i) Δ is continuous since each f_n is continuous and $p_n \circ \Delta = f_n$.

(ii) Δ is injective:

Let $x, y \in X$ with $x \neq y$.

Since X is T_1, $\cap\{S \mid x \in S \in \mathcal{S}\} = \{x\}$ and hence there exists $S_0 \in \mathcal{S}$ such that $x \in S_0$ and $y \notin S_0$.

Suppose $S_0 \in E_n$. By Lemma 1 and uniqueness of E'_n $\exists S'_0 \in E'_n$ such that $y \in S'_0$ and $S_0 \cap S'_0 = \Phi$.

Now $S_0 = f^{-1}(-\infty, a]$, $a \in Q$, so that $f_n(x) \leqq a$.

Since $S_0 \cap S'_0 = \Phi$, $f_n(y) > a$ and hence $f_n(x) \neq f_n(y)$.

Thus $\Delta(x) \neq \Delta(y)$.

If $S_0 \in E'_n$ a similar proof, using Corollary (ii)(b), holds.

(iii) Δ is a surjection:

Let $(a_n)_{n \in A} \in \Pi\{R_n \mid n \in A\}$. Consider

$$\mathcal{F} = \left|\bigcup_{n \in A}\{S \in E_n \mid f_n^{-1}(-\infty, a_n] \subset S\}\right] \cup \left|\bigcup_{n \in A}\{S' \in E'_n \mid f_n^{-1}[a_n, \infty) \subset S'\}\right| .$$

Clearly $\mathcal{F} \cap E \neq \Phi$ and $E \not\subset \mathcal{F}$ $\forall E \in \mathcal{E}$.

Furthermore in view of Lemma 8, $\mathcal{F}$ is linked and thus by condition (6) $\cap \mathcal{F} \neq \Phi$.

Let $x \in \cap \mathcal{F}$. We show that $\Delta(x) = (a_n)_{n \in A}$, i.e. $f_n(x) = a_n$ $\forall n \in A$.

Suppose not, i.e. suppose $\exists n \in A$ such that $f_n(x) \neq a_n$.

(a) Suppose $a_n < f_n(x)$ and choose $q \in Q$ such that $a_n < q < f_n(x)$.

By Lemma 7, $S = f_n^{-1}(-\infty, q] \in E_n$ and since $f_n^{-1}(-\infty, a_n] \subset S$ it follows that $S \in \mathcal{F}$.

However, $x \notin f_n^{-1}(-\infty, q]$, a contradiction.

(b) Suppose $a_n > f_n(x)$.

By Lemma 7, Corollary (ii)(c) there exists $S' = f_n^{-1}[b, \infty) \in E'_n$ for some $b \in R$ satisfying $a_n > b > f_n(x)$.

Since $f_n^{-1}[a_n, \infty) \subset S'$ $S' \in \mathcal{F}$.

However, $x \notin f_n^{-1}[b, \infty)$, a contradiction.

Thus $f_n(x) = a_n$ $\forall n \in A$.

(iv) Δ^{-1} is continuous:

We keep in mind that Δ is a bijection and prove that $\Delta(S)$ is closed in $\Pi\{R_n \mid n \in A\}$ for all $S \in \mathcal{S}$.

Now $\forall S \in \mathcal{S}$ $\exists n \in A$ such that $S = f_n^{-1}(-\infty, r]$ or $S = f_n^{-1}[a, \infty)$ for some $r \in Q$ or some $a \in R$.

Suppose $S = f_n^{-1}(-\infty, r]$. We prove that $\Delta(S) = p_n^{-1}(-\infty, r]$:

Let $y \in \Delta(S)$, then $y = \Delta(x)$ for some $x \in S$ and since $f_n(x) \leqq r$, $p_n \circ \Delta(x) \leqq r$ so that $y \in p_n^{-1}(-\infty, r]$.

Let $z \in p_n^{-1}(-\infty, r]$. Now $\exists x \in X$ such that $\Delta(x) = z$. If $x \notin S = f_n^{-1}(-\infty, r]$ then $f_n(x) > r$, i.e. $p_n(z) > r$, a contradiction.

Thus $x \in S$ and hence $z = \Delta(x) \in \Delta(S)$.

Hence $\Delta(S) = p_n^{-1}(-\infty, r]$.

If $S = f_n^{-1}[a, \infty)$ then it follows in a similar way that $\Delta(S) = p_n^{-1}[a, \infty)$.

The proof of the Theorem is now complete.

Remark. If there are a countable infinite number of equivalence classes, $X \cong R^{\infty}$ and hence in view of Anderson's result ([1] and [2]), $X \cong l_2$.

If a specific characterization of Hilbert space is sought one could require the additional condition:

(7) $|\{E\}|$ is not finite.

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