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Polar reciprocal Simplexes

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Geometria proiettiva. — *Polar reciprocal Simplexes.* Nota di AUGUSTINE O. KONNULLY, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra che, dati in uno spazio proiettivo n–dimensionale una quadrica Q ed un simplesso S i cui vertici siano a due a due non coniugati rispetto a Q, le $n(n+1)/2$ coppie di ciascun spigolo rettilineo di S coniugate di un vertice rispetto a Q giacciono su di un'altra quadrica. Questa ed S definiscono Q in $2^{n(n+1)/2}$ modi diversi; il che (per $n > 2$) costituisce un'estensione del classico Teorema di Pascal sull'esagono inscritto in una conica.

1. INTRODUCTION

Given a pair of simplexes in n–space, polar reciprocal for a quadric, so that the prime faces of one are the polars of the vertices of the other simplex, they are, in general, not perspective. But the joins of the corresponding vertices of the two simplexes form an associated set of $(n+1)$ lines such that any $(n-2)$–space which meets n of them meets the remaining one also [1, 2]; in the case $n = 2$ this means that the point which lies on two of the lines lies on the third also, that is, the lines concur.

It has been shown that, when two simplexes are perspective from a point, the traces of the prime faces of one simplex on the edges of the other, other than those falling on the prime of perspectivety lie on a quadric [3, 4, 5]. Here we prove a similar result, polar reciprocal simplexes replacing perspective simplexes; which will be a generalisation of the above result on perspective simplexes. The converse which also will be proved, will be a generalisation and extension of Pascal's theorem regarding the pair of triangles formed by six points on a conic.

2. ASSOCIATED QUADRICS

THEOREM. *Let* S *and* T *be two simplexes in n–space such that one is the polar reciprocal of the other with respect to a quadric; and suppose that no two vertices of* S *are conjugate with respect to this quadric. Then the traces of the prime faces of* S *on the edges of* T, *other than those falling on the intersection of corresponding primes of the two simplexes, lie on another quadric.*

Proof. Let S be the simplex of reference with $A_0, A_1, \cdots, A_n$ for vertices and $a^0, a^1, \cdots, a^n$ for prime faces. Let $B_0, B_1, \cdots, B_n$ be the vertices and $b^0, b^1, \cdots, b^n$, the prime faces of T. Let

$$\sum_{i,j=0}^{n} g_{ij}\, x_i\, x_j = 0\,, \qquad (g_{ij} = g_{ji}) \tag{1}$$

(*) Nella seduta del 12 febbraio 1972.

be the quadric with respect to which S and T are polar reciprocal; from our assumption on S it follows that $g_{ij} \neq 0$ for $i \neq j$. Being the polar of A_i with respect to the quadric (1), the prime face b^i of T will be

$$\sum_{j=0}^{n} g_{ij} x_j = 0 , \qquad (i = 0 , 1 , 2 , \cdots , n) . \tag{2}$$

Consider the traces of each b^i on the edges of S, other than those lying in the corresponding face a^i of S. Let P_{ij}, $j = 0 , 1 , \cdots , i - 1 , i + 1 , \cdots , n$, be the traces of b^i on the n edges $A_i A_j$ which do not lie in a^i. There are $n(n + 1)$ such points, a pair on each edge, P_{ij} and P_{ji} on $A_i A_j$.

The $n(n + 1)$ points P_{ij} lie on a quadric for the point P_{ij} satisfies the equation $g_{ij} x_j + g_{ii} x_i = 0$, and P_{ji} the equation $g_{ji} x_i + g_{jj} x_j = 0$ (with $x_k = 0$ for $k \neq i , j$). Hence the pair of points satisfy the equation

$$(g_{ii} x_i + g_{ij} x_j) (g_{jj} x_j + g_{ij} x_i) = 0 ,$$

that is,

$$g_{ii} x_i^2 + \frac{1}{g_{ij}} (g_{ij}^2 + g_{ii} g_{jj}) x_i x_j + g_{jj} x_j^2 = 0 ; \qquad x_k = 0 \quad , \quad k \neq i , j .$$

So the quadric,

$$\sum_{i,j=0}^{n} c_{ij} x_i x_j = 0 , \qquad \text{where} \quad c_{ij} = \frac{1}{2} (g_{ij}^2 + g_{ii} g_{jj}) / g_{ij} , \tag{3}$$

passes through the points P_{ij} and P_{ji} $(i , j = 0 , 1 , \cdots , n , i \neq j)$.

Dually, if q^{ij} is the hyperplane joining B_i to the $(n - 2)$–dimensional face of S which is the intersection of a^i and a^j, the $n(n + 1)$ hyperplanes, we will have thus, a pair through each $(n - 2)$–face of S, namely, q^{ij} and q^{ji} through $a^i \cdot a^j$, all touch a quadric.

This quadric is evidently the reciprocal with respect to (1) of the quadric through the points Q_{ij} which are the traces of the prime faces of S on the edges of T.

3. THE CONVERSE

THEOREM. *Let a quadric meet the edges of a simplex* $S = A_0 A_1 \cdots A_n$ *in pairs of points; let* P_{ij} *and* P_{ji} *be, in any fixed order, the points in which it meets the edge* $A_i A_j$. *If* p^i *is the prime containing the n points* $P_{i0} , P_{i1} , \cdots , P_{in}$ *which lie each on one edge through the vertex* A_i, $(i = 0 , 1 , \cdots , n)$, *then the simplex* T *with* $p^0, p^1, \cdots , p^n$ *for prime faces is the polar reciprocal of* S *with respect to a quadric.*

Proof. Let S be taken as the simplex of reference. Let $\sum c_{ij} x_i x_j = 0$ be the quadric which meets the edges of S. The points P_{ij} and P_{ji} are then given by

$$c_{ii} x_i^2 + 2 c_{ij} x_i x_j + c_{jj} x_j^2 = 0 ,$$

that is,

$$\{c_{ii}\,x_i + (c_{ij} + \sqrt{c_{ij}^2 - c_{ii}\,c_{jj}}\,)\,x_j\}\,\{c_{ii}\,x_i + (c_{ij} - \sqrt{c_{ij}^2 - c_{ii}\,c_{jj}}\,)\,x_j\} = 0$$

(with $x_k = 0$ for $k \neq i\,,\,j$). So the point P_{ij} will be

$$x_k = 0 \quad , \quad k \neq i\,,\,j\,; \qquad c_{ii}\,x_i + (c_{ij} + \theta_{ij}\sqrt{c_{ij}^2 - c_{ii}\,c_{jj}}\,)\,x_j = 0\,,$$

where $\theta_{ij} = +1$ or -1 and the point P_{ji} then is

$$x_k = 0 \quad , \quad k \neq i\,,\,j\,; \qquad c_{ii}\,x_i + (c_{ij} - \theta_{ij}\sqrt{c_{ij}^2 - c_{ii}\,c_{jj}}\,)\,x_j = 0\,,$$

which may be equivalently written as

$$c_{jj}\,x_j + (c_{ji} + \theta_{ji}\sqrt{c_{ij}^2 - c_{ii}\,c_{jj}})\,x_i = 0\,, \qquad \theta_{ij} = \theta_{ji}\,.$$

So, the prime

$$\sum_{j=0}^{n}(c_{ij} + \theta_{ij}\sqrt{c_{ij}^2 - c_{ii}\,c_{jj}})\,x_j = 0\,, \quad (\theta_{i0} = \pm 1\,,\ \theta_{i1} = \pm 1\,,\cdots,\theta_{in} = \pm 1)$$

contains all the points $P_{i0}\,,\,P_{i1}\,,\cdots,\,P_{in}$. Thus the primes $p^0, p^1,\cdots, p^n$ are

$$\sum_{j=0}^{n}(c_{ij} + \theta_{ij}\sqrt{c_{ij}^2 - c_{ii}\,c_{jj}})\,x_j = 0\,, \qquad (i = 0\,,\,1\,,\cdots,\,n)\,, \tag{4}$$

where each $\theta_{ij} = \pm 1$ and $\theta_{ij} = \theta_{ji}$ for each i and j.

And these are evidently the polars of the vertices of the simplex of reference with respect to the quadric

$$\sum_{i,j=0}^{n} g_{ij}\,x_i\,x_j = 0\,, \qquad \text{where} \quad g_{ij} = c_{ij} + \theta_{ij}\sqrt{c_{ij}^2 - c_{ii}\,c_{jj}}\,, \tag{5}$$

($g_{ij} = g_{ji}$, since $c_{ij} = c_{ji}$ and $\theta_{ij} = \theta_{ji}$).

That is, the simplexes S and T are polar reciprocal for (5).

The $n\,(n+1)$ points P_{ij} in which the quadric meets the edges of S fall by sets of n in $(n+1)$ primes p^i such that the n points of every prime p^i lie on the n concurrent edges of S through a vertex A_i, one on each edge; and the $(n+1)$ primes p^i form a simplex T. There are $2^{n(n+1)/2}$ ways of the $n(n+1)$ points distributing in this manner there being two choices for either of the two points on every edge independent of one another. Corresponding to each way of distribution there is a simplex T which is polar reciprocal to S with respect to a quadric. These simplexes are easily seen to be the polar reciprocals of S with respect to the $2^{n(n+1)/2}$ different quadrics $\sum(c_{ij} + \theta_{ij}\sqrt{c_{ij}^2 - c_{ii}\,c_{jj}})\,x_i\,x_j = 0$ we obtain on assigning different values to $\theta_{ij} = \pm 1$.

In 2–dimensions, given six points $P_1\,,\,P_2\,,\cdots,\,P_6$ on a conic, these will be the points in which the conic meets the triangle formed by the lines $P_1\,P_2\,,\ P_3\,P_4\,,\ P_5\,P_6$. And Pascal's theorem states that this triangle and the triangle formed by $P_2 P_3$, $P_4 P_5$, $P_6 P_1$ are perspective from a line; this may be expressed as well by saying that the two triangles are polar reciprocals with respect to a conic, since for two triangles to be reciprocals with respect to a conic is the same as to be perspective.

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