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On the asymptotic distribution of values of arithmetical functions

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Teoria dei numeri. — *On the asymptotic distribution of values of arithmetical functions.* Nota di JÁNOS GALAMBOS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota si studia la distribuzione asintotica di funzioni aritmetiche non necessariamente additive o moltiplicative.

INTRODUCTION

We say that the arithmetical function $f(n)$ has a limit distribution $F(x)$ if, for all continuity points of $F(x)$,

$$\lim_{N \Rightarrow +\infty} \frac{1}{N} \sum_{\substack{n \leq N \\ f(n) < x}} 1 = F(x) . \tag{1}$$

Putting

$$F_N(x) = \frac{1}{N} \sum_{\substack{n \leq N \\ f(n) < x}} 1 , \tag{2}$$

(1) can be rewritten as $\lim F_N(x) = F(x)$ as $N \Rightarrow +\infty$. One frequently used technique to find conditions on $f(n)$ for (1) to hold is to turn to the Fourier transforms of $F_N(x)$ and $F(x)$. The power of the Fourier transforms lies in the continuity theorem which enables one to replace the relation (1) by the limit of the corresponding Fourier transforms. More precisely, setting

$$\varphi_N(t) = \frac{1}{N} \sum_{n=1}^{N} e^{it f(n)} \tag{3}$$

and

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} \, dF(x) , \tag{4}$$

we have (see Loéve [9], p. 191) the following

CONTINUITY THEOREM. *If* (1) *holds, then* $\varphi_N(t) \to \varphi(t)$. *Conversely, if* $\lim \varphi_N(t)$ *exists, as* $N \Rightarrow +\infty$, *and is continuous at* $t = 0$, *then* (1) *holds, and lim* $\varphi_N(t) = \varphi(t)$, $N \to +\infty$.

In the past decade, much attention was paid to the distribution problem of $f(n)$ for the special case when $f(n)$ is additive; i.e., when, for $(u, v) = 1$, $f(uv) = f(u) + f(v)$. The special characteristic of this class of arithmetical

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functions is that they behave as sums of (asymptotically) independent random variables. Some results can therefore be obtained by purely probabilistic arguments (see Galambos [5]), but at least most results can be guessed from similar theorems of probability theory and then by a unified method they can actually be proved (see Kubilius [8]). The method of power series, which can be labelled as Delange's method (see e.g. Delange [2]), is also very powerful in this field.

The class of multiplicative functions $g(n)$, i.e., such that, for $(u, v) = 1$, $g(uv) = g(u)g(v)$, is strongly related to the additive ones, since there is a one to one correspondence between the additive and positive multiplicative functions by the relation $\exp(f(n)) = g(n)$. Hence the distribution problem of multiplicative functions is more general and, in certain sense, more difficult than that of additive functions. General solutions for these were obtained only recently by Bakstys [1] and Galambos [6].

For arithmetical functions, when neither additivity nor multiplicativity is assumed, very little is known. A general Theorem in this direction was independently obtained in Novoselov [10] and Galambos [4]. In the present paper we shall extend this result and give some examples which in themselves are of interest. In a series of papers, further extensions will be given. In addition to the Continuity Theorem, properties of Dirichlet series will be made use of.

THE RESULT

We introduce the following notation. Let $f(n)$ be an arbitrary arithmetical function. Then

$$\text{(5)} \qquad \mathrm{F}_f(s, t) = \sum_{n=1}^{+\infty} \frac{e^{it f(n)}}{n^s} .$$

There are well known methods to express $\varphi_{\mathrm{N}}(t)$ of (3) in terms of $\mathrm{F}_f(s, t)$, though we shall use them in an indirect way. Our main result is the following

THEOREM. *Let $f(n)$ be an arbitrary arithmetical function and let $g(n)$ be such that its asymptotic distribution exists. Assume that $\mathrm{F}_f(s, t)/\mathrm{F}_g(s, t) = \mathrm{H}(s, t)$ can be expanded into a Dirichlet series which is absolutely convergent at $s = 1$. Then, if $\mathrm{H}(s, t)$ as a function of t is continuous at $t = 0$, the asymptotic distribution of $f(n)$ also exists. The Fourier transform of the asymptotic distribution of $f(n)$ is $\mathrm{H}(1, t)$ times the Fourier transform of the limit distribution of $g(n)$.*

The proof is based on the Continuity Theorem and on the following Lemma, which, in several specific forms, has been applied by many authors and which seems to be due to Wintner. A detailed proof is given in Rényi [11].

LEMMA. *Let us assume that*

$$\sum_{n=1}^{+\infty} \frac{|b_n|}{n} \quad \textit{and} \quad \frac{1}{N}\sum_{n=1}^{N} c_n \tag{6}$$

converge and that

$$\sum_{n=1}^{+\infty} \frac{a_n}{n^s} = \sum_{n=1}^{+\infty} \frac{b_n}{n^s} \sum_{n=1}^{+\infty} \frac{c_n}{n^s} . \tag{7}$$

Then

$$\lim_{N \Rightarrow +\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = \sum_{n=1}^{+\infty} \frac{b_n}{n} \lim_{N \Rightarrow +\infty} \frac{1}{N} \sum_{n=1}^{N} c_n .$$

Proof of the Theorem. Let

$$H(s, t) = \sum_{n=1}^{+\infty} \frac{h(n,t)}{n^s} .$$

We then apply the Lemma with the following choices:

$$a_n = e^{itf(n)} , \ b_n = h(n, t) \quad \text{and} \quad c_n = e^{it\,g(n)} .$$

By the assumption on $g(n)$ and by the continuity Theorem, the sequence in (6) converges and the limit is a characteristic function, therefore it is continuous at any t, hence at $t = 0$. The convergence of the series in (6) and the validity of (7) follow from our conditions and thus the conclusion of the Lemma yields that $\varphi_N(t)$, defined in (3), has a limit as $N \Rightarrow +\infty$, and that the limit is $H(1, t)$ times the limit of the sequence of (6). Since both factors are continuous at $t = 0$, the second part of the Continuity Theorem terminates the proof.

Though the proof is an easy combination of the Continuity Theorem and the Lemma, the following example, contained in a Corollary, shows its significance.

Let $B_1, B_2, \cdots$, be a sequence of positive integers and assume that, if $U(m)$ denotes the number of distinct prime factors of m,

$$U(B_j) \leq r , j = 1, 2, \cdots, \quad r \text{ fixed,} \tag{8}$$

and that

$$\sum_{j=1}^{+\infty} \frac{1}{B_j} < +\infty . \tag{9}$$

Suppose that no B_j contains squarefree part (i.e. all prime divisors of B_j are at least of the second degree), and let $A_1, A_2, \cdots$, be the sequence of those positive integers which can be written in the form $B_j Q$ with some j and where Q is a squarefree number. By our assumption, for any m, the representation $A_m = B_j Q$ is unique. Let us define the arithmetical function $f(n)$ as

$$f(n) = \begin{cases} v(B_j) + g(Q) & \text{if for some } m, n = A_m = B_j Q \\ g(n) & \text{otherwise,} \end{cases} \tag{10}$$

where $v(a)$ is an arbitrary arithmetical function and $g(a)$ is strongly additive (i.e. additive and for $\alpha \geq 1$, $g(p^\alpha) = g(p)$ for any prime number p). We shall show, as a corollary to our Theorem, that if $g(n)$ has an asymptotic distribution so does $f(n)$, independently of the function $v(a)$. But since $f(n)$ is not additive, this example shows that not so much the additivity as the asymptotic behaviour of the Dirichlet series $F_g(s, t)$, or equivalently, $F_f(s, t)$, is the determining factor in the existence of the asymptotic law for either g or f. Note that $f(n)$ differs from $g(n)$ on a large set $A_1, A_2, \cdots$ (which has positive asymptotic density).

COROLLARY. *Let $f(n)$ be defined by* (10). *Assume that each of the series*

$$\sum_{|f(p)|<1} \frac{f(p)}{p} \quad , \quad \sum_{|f(p)|<1} \frac{f^2(p)}{p} \quad , \quad \sum_{|f(p)|\geq 1} \frac{1}{p} \tag{11}$$

converges where p runs over the prime numbers. Then the limit distribution of $f(n)$ exists.

Proof. Note that for any prime p, $f(p) = g(p)$. (11) thus reduces to the Erdös [3] Theorem on $g(n)$ which yields that the asymptotic distribution of $g(n)$ exists. In order to apply our Theorem, we therefore have to evaluate the ratio $F_f(s, t)/F_g(s, t)$. By definition

$$F_f(s, t) = \sum_{m=1}^{+\infty} \frac{e^{it[v(B_j)+g(Q)]}}{A_m^s} + \sum_{n \neq A_m} \frac{e^{itg(n)}}{n^s} =$$

$$= \sum_{j=1}^{+\infty} \sum_{Q} \frac{[e^{itv(B_j)} - e^{itg(B_j)}]\, e^{itg(Q)}}{B_j^s Q^s} + \sum_{n=1}^{+\infty} \frac{e^{itg(n)}}{n^s},$$

where Q is squarefree and $A_m = B_j Q$ is the unique representation in the definition of A_m. We therefore have

$$F_f(s, t) = \prod_{p} \left(1 + \frac{e^{itg(p)}}{p^s}\right) \sum_{j=1}^{+\infty} \frac{e^{itv(B_j)} - e^{itg(B_j)}}{B_j^s} \prod_{p \mid B_j} \left(1 + \frac{e^{itg(p)}}{p^s}\right)^{-1} + F_g(s, t).$$

By $g(n)$ being strongly additive,

$$F_g(s, t) = \prod_{p} \left(1 + \frac{e^{itg(p)}}{p^s - 1}\right).$$

Thus, since

$$\left(1 + \frac{e^{itg(p)}}{p^s}\right) \Big/ \left(1 + \frac{e^{itg(p)}}{p^s - 1}\right) = 1 - \frac{e^{itg(p)}}{p^s(p^s - 1)} \left(1 + \frac{e^{itg(p)}}{p^s - 1}\right)^{-1},$$

our assumptions (8) and (9) on the sequence B_j imply that $F_f(s, t)/F_g(s, t)$ is absolutely and uniformly convergent at $s = 1$ and it is therefore continuous in t. The conclusion of our Theorem thus establishes the Corollary.

A few remarks are in order. Note that no assumption is made on the function $v(a)$ in the Corollary, which means that the function $f(n)$ is even not determined by the terms occuring in (11), which guarantees the existence of its asymptotic distribution. The choices of $g(n) = 0$ and $v(a) = 1$

are of special interest, when $f(A_m) = 1$ for all m and zero for other values of the argument. Since now $f(p) = g(p) = 0$ for all primes, (11) is satisfied; thus, in view of (1) and (2), the Corollary implies, as a special case, that the sequence A_m has positive density in the set of successive integers. This explains our earlier claim that $f(n)$ is not additive on a 'large' set. This same fact raises an interesting question. Is there any arithmetical function $f(n)$ which is not additive on a set of density one and at the same time (11) guarantees the existence of its asymptotic distribution? There is, of course, a trivial solution to this question, namely, if $f(n) = g(n) + 2$, say, if n is not a prime number and $f(p) = g(p)$ for all primes, where $g(n)$ is a strongly additive function. The question would be interesting even if we replace "density" by "upper density".

EXAMPLES

Let the sequence $B_1, B_2, \cdots$ be defined as the set of those positive integers which have the canonical representation $p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_k}^{\alpha_k}$ with $\alpha_{i_t} \geq 2$, $t = 1, 2, \cdots, k$, and for which $(\alpha_{i_1} - 1) + (\alpha_{i_2} - 1) + \cdots + (\alpha_{i_k} - 1) = r$, a given number. (8) is evidently satisfied, and since

$$\sum_{j=1}^{+\infty} \frac{1}{B_j} \leq \left(\sum_p \frac{1}{p^2}\right)^r + \left(\sum_p \frac{1}{p^3}\right)^{r-1} + \cdots < +\infty,$$

so is (9). The Corollary is therefore applicable, which provides a great freedom to modify a strongly additive function $g(n)$ on the set A_m, the set of those integers for which the difference between the numbers of all prime divisors and the distinct prime divisors is a fixed number r, and yet (11) guarantees the existence of the asymptotic distribution. It is well known that the set A_m has positive density (see Galambos [5] for several references), which fact is reobtained in the Corollary by a special choice of $f(n)$, as described in the previous section.

The Corollary itself actually gives a wide class of examples, hence we turn to additional applications of the Theorem. First of all, let us point out that, choosing $g(n) = 0$ for all n, $F_g(s, t)$ reduces to Riemann's zeta function; thus our Theorem does extend the result of Galambos [4] and the corresponding theorem in Novoselov [10]. Without going into details, we wish to indicate two classes of examples.

One is to emphasize that the Theorem can be applied to sequences of arithmetical functions if, for one of the sequences, the asymptotic distribution is known. Let, for example, $g(n) = (U(n) - \log\log N)/(\log\log N)^{1/2}$, where, as before, $U(n)$ is the number of distinct prime divisors of n. Then, with $f(n)$ satisfying the conditions in the Theorem, a limit distribution for $(f(n) - E_N)/D_N$ with suitable sequences E_N and D_N of real number can be obtained. As a matter of fact, for $g(n)$ specified above it is known that its Fourier transform corresponding to (3) tends to $\exp\left(-\frac{1}{2}t^2\right)$, as $N \Rightarrow +\infty$. Several 'normalized' arithmetical functions can be 'compared' by our result.

The final class of examples is rather a remark. Kátai and Mogyoródi [7] introduced the following class of arithmetical functions. Let n be represented as a product of a complete square and a squarefree number, $n = SQ$, say. It is immediately seen that this representation is unique. They consider those arithmetical functions which are functions of S only (in my reformulation here, S and Q are not relatively prime, but evidently equivalent to their definition). Several of their neat results can be reobtained from our Theorem by considering first a specific form of their functions, we take it as $g(n)$, and then extend the result through our Theorem to a more general form $f(n)$.

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