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Algebra topologica. — *Function Algebras over Valued Fields and Measures.* Nota II di GEORGE BACHMAN, EDWARD BECKENSTEIN e LAWRENCE NARICI, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Si studia l'algebra topologica F(T) delle funzioni continue che applicano uno spazio O–dimensionale T in un campo valutato non archimedeo completo, munito della topologia compatto–aperta.

3. ZERO-ONE MEASURES

In this section T is a O–dimensional Hausdorff space. Prior to Definition 3, F may be any field whose characteristic is O; in Lemma 1 and all subsequent results F is a nonarchimedean nontrivially valued field of characteristic O to which $\sqrt{-1}$ does not belong. (See Sec. 4).

DEFINITION 1. A *zero-one measure* is a map μ of the clopen sets $\mathcal{S}$ of T into $\{O, 1\} \subset F$ such that: (*a*) for any two disjoint clopen sets U and V, $\mu(U \cup V) = \mu(U) + \mu(V)$; (*b*) $\mu(T) = 1$.

Clearly if μ is a zero-one measure, μ is a bounded measure and $\mathfrak{D}_\mu = \{U \in \mathcal{S} \mid \mu(U) = 0\}$. Thus $F_\mu = C(\cup \mathfrak{D}_\mu) = \cap \{U \in \mathcal{S} \mid \mu(U) = 1\}$. Moreover, if μ is a zero-one measure, F_μ is compact as we now prove. To this end, suppose that F_μ is nonempty, (U_s) is a clopen cover of F_μ, and $F_\mu \cap U_t \neq \varnothing$ for some index t. If $s \neq t$ then $F_\mu \cap (U_s - U_t) = \varnothing$: otherwise $\mu(U_s - U_t) = \mu(U_t) = 1$ and $\mu((U_s - U_t) \cup U_t) = 2$. Thus it follows that $F_\mu \subset U_t$, so F_μ is compact.

Actually much more is true of F_μ for zero-one measures. The above argument shows that if (U_s) is a clopen cover of F_μ, then F_μ must be wholly contained in any U_s which it meets. Since T is Hausdorff it follows that if F_μ is not empty, then it must be a singleton. This proves:

PROPOSITION 6. *If μ is a zero-one measure, then its support F_μ is empty or is a singleton.*

DEFINITION 2. A zero-one measure μ is *concentrated at a point t* if there is some $t \in T$ such that $\mu(U) = 1$ if and only if $t \in U$.

As an immediate consequence of Proposition 6 we have:

PROPOSITION 7. *A zero-one measure μ is concentrated at a point if and only if F_μ is not empty.*

DEFINITION 3. Let F be a valued field so that we can consider the algebra F(T) of *continuous* functions mapping T into F. For $H \subset F(T)$, $Z(H) = \bigcup_{f \in H} f^{-1}(O)$. A *z-filter* on T is a nonempty subfamily $\mathcal{F}$ of $Z(F(T))$

(*) Nella seduta dell'11 dicembre 1971.

which does not have $\varnothing$ as a member, is closed with respect to the formation of finite intersections, and is such that if $S \in \mathfrak{F}$ is contained in $S' \in Z(F(T))$, then $S' \in \mathfrak{F}$. A *z-ultrafilter* is a *z*–filter which is not properly contained in any other *z*–filter.

In what follows F is a nonarchimedean nontrivially valued field of characteristic O which does not contain $\sqrt{-1}$ (such as, for example, the *p*–adic number fields Q_p for $p = 4n + 3$, $n = 1, 2, \cdots$; cf. [1, p. 59]), F(T) carries the compact-open topology and M denotes a maximal ideal in F(T).

LEMMA 1. *If* M *is a maximal ideal in* F(T), *then* Z(M) *is a z-ultrafilter.*

Proof. If $\varnothing \in Z(M)$ there would be an $f \in M$ such that $f^{-1} \in M$; hence $\varnothing \notin Z(M)$. If $S, W \in Z(M)$ then $S = f^{-1}(O)$ and $W = g^{-1}(O)$ for some $f, g \in M$. Since $\sqrt{-1} \notin F$, $S \cap W = Z(f) \cap Z(g) = Z(f^2 + g^2)$; thus $S \cap W \in Z(M)$. Suppose now that $S = f^{-1}(O) \in Z(M)$ and $S \subset W = g^{-1}(O) \in Z(F(T))$. Letting $h = fg$, then $W = h^{-1}(O) \in Z(M)$.

To show that Z(M) is a *z*–ultrafilter, let $S \in Z(F(T))$ and suppose that $S \notin Z(M)$; thus $S = f^{-1}(O)$ where $f \notin M$. Consequently the ideal generated by M and f is F(T) so there must be functions $g \in F(T)$ and $h \in M$ such that $fg + h = 1$. Thus $\varnothing = Z(fg + h) \supset Z(f) \cap Z(h) = S \cap Z(h)$. Hence there can be no *z*–filter to which S belongs that contains Z(M). Hence Z(M) is a *z*–ultrafilter.

THEOREM 4. *Associated with each maximal ideal* M *in* F(T) *is a zero-one measure* μ. *The measure* μ *is concentrated at a point if and only if* M *is the kernel of an evaluation map* $\hat{t}$ *where* $\hat{t}(f) = f(t)$ *for all* $f \in F(T)$. *In this case* $F_\mu = \{t\}$.

Proof. We let $U \in \mathfrak{S}$ and set $\mu(U) = 1$ if $U \supset S$ for some $S \in Z(M)$. Otherwise $\mu(U) = 0$. Equivalently, as $U \in Z(F(T))$, $\mu(U) = 1$ if and only if $U \in Z(M)$.

We show that μ satisfies condition (*a*) of Definition 1. Let $W \in S$ and suppose $U \cap W = \varnothing$. Clearly both U and W cannot belong to Z(M). If $U \in Z(M)$ and $W \notin Z(M)$, then clearly $\mu(U \cup W) = \mu(U) + \mu(W) = 1$. As Z(M) is a *z*–ultrafilter and $U \cup W \in Z(F(T))$, then if both U and W do not belong to Z(M), $U \cup W$ also does not belong to Z(M). Thus $\mu(U \cup W) = \mu(U) + \mu(W) = 0$.

Since $F_\mu = \bigcap_{\mu(U)=1} U = \bigcap_{U \in Z(M)} U$ and T is O–dimensional and Hausdorff, then $F_\mu = \bigcap_{f \in M} Z(f)$. Thus $F_\mu = \{t\}$ if and only if all functions in M vanish at t. Thus M is the kernel of the evaluation map $\hat{t}$.

DEFINITION 4. T is an F — Q space if and only if the zero-one measures generated by the kernels of nontrivial homomorphisms of F(T) into F are concentrated at points. (See Sec. 4).

Note that what is being asked of T in Definition 4 is that the homomorphisms of F(T) into F be the evaluation maps $\hat{t}$ for all $t \in T$.

THEOREM 5. *Let* T *be a Lindelöf space. Then* T *is an* F — Q *space.*

Proof. Let J be a nontrivial homomorphism of F(T) into F and suppose that F_{μ_J} is empty where μ_J is the zero-one measure associated (Theorem 4) with the kernel of J. Then there exists a family (W_j) of pairwise disjoint clopen sets such that $T = \bigcup_{i=1}^{\infty} W_i$ where each $\mu_J(W_i) = 0$. Define a function

$$f = \sum_{i=1}^{\infty} \pi^i k_{W_i}$$

where $|\pi| < 1$. The function $f \in F(T)$ and therefore for some $\lambda \in F$, $f - \lambda e \in M$ where M is the kernel of J. Thus $f - \lambda e$ cannot be invertible in F(T) and therefore for some $t \in T$, $(f - \lambda e)(t) = 0$. Then clearly $\lambda = \pi^i$ for some i and $(f - \lambda e)^{-1}\{O\} = W_i$. Hence $\mu_J(W_i) \neq 0$ and we have arrived at a contradiction.

By a Theorem of Van Rooij [12, p. 27,29] it follows that T is homeomorphic to a closed subspace of a product of the field F.

In [3] we proved an analog of a Theorem of S. Warner [15, p. 268] in the nonarchimedean setting. The statement of this Theorem is as follows:

Let the set of nontrivial continuous homomorphisms on the Fréchet full algebra X over a local field F be denoted by $\mathfrak{M}$. Let $\mathfrak{M}$ carry the weak-* (Gelfand) topology and $F(\mathfrak{M})$ the compact-open topology. Then X is topologically isomorphic to $F(\mathfrak{M})$.

We observe that if F is a local field without $\sqrt{-1}$, then the algebra X of the above-quoted Theorem is seen to be a functionally continuous algebra after it is observed that $\mathfrak{M}$ is a Lindelöf space and the preceeding Theorem is applied.

Example 1. If T is an uncountable set of nonmeasurable cardinal carrying the discrete topology and F is discretely valued, then T is an F — Q space, but clearly not a Lindelöf space.

Example 2. We let F be a local field (once again of characteristic O and not containing $\sqrt{-1}$). We choose our space T as the field itself and take the topology on T to be induced by the valuation. Then T is a Lindelöf space. Thus the homomorphisms of F(T) are the points of T by Theorem 5.

We let $A_n = \{t \mid |t| > n\}$ $(n \geq 1)$ and $I = \{f \mid f(A_n) = \{0\}$ for some $n\}$. I is an ideal in F(T) and by Zorn's lemma can be extended to a maximal ideal. Of course I is the ideal of functions with compact support in F(T). Since the sets A_n are clopen in T, the functions k_{CA_n} are in I. Since each $t \in T$ belongs to CA_n for some n and k_{CA_n} is equal to 1 at t, M cannot be the kernel of a homomorphism of F(T) into F. Then associated with M is a zero-one measure μ with empty support.

If we take μ_0 to be the zero-one measure concentrated at $t = 0$ (associated with the evaluation map $\hat{t} = \hat{0}$) we see that $\hat{\mu} = \mu + \mu_0$ is a bounded measure with compact support having the following properties

(*a*) $F_{\hat{\mu}} = \{0\}$;

(*b*) $\hat{\mu}$ restricted to $F_{\hat{\mu}}$ is not well-defined;

(*c*) $\hat{\mu}$ does not satisfy condition [M] of [13, p. 190];

(*d*) The linear functional h defined by;

$$h\left(\sum_{i=1}^{n} \alpha_i k_{W_i}\right) = \sum_{i=1}^{n} \alpha_i \hat{\mu}(W_i)$$

on $X = [k_W \mid W \in \mathcal{S}]$ is not continuous.

Proof. To show (*b*) is true we observe that $CA_n \cap F_{\hat{\mu}} = F_{\hat{\mu}}$ and $T \cap F_{\hat{\mu}} = F_{\hat{\mu}}$. However $\hat{\mu}(CA_n) = 1$ while $\hat{\mu}(T) = 2$.

To prove (*c*) simply note that $\bigcap_{n=1}^{\infty} A_n = \emptyset$ but $\hat{\mu}(A_n) = 1$ and does not converge to O.

The remaining statements have already been proved in previous discussions and the example is finished.

We consider a final application of zero-one measures. In this discussion we assume that F is a discretely valued field (of characteristic O). We do not insist that $\sqrt{-1}$ not be present in F. In [3] it is shown through use of the theory of nonarchimedean locally multiplicatively convex topological algebras that when F satisfies the conditions imposed above, the continuous homomorphisms of F(T) into F (again F(T) carries the compact-open topology) are the evaluation maps on the points of T. We now prove this statement using zero-one measures.

Let h be a continuous homomorphism and μ_h the measure associated with h by the results of Sec. 2 of this paper. Then letting μ_h^* be the restriction of μ_h to F_{μ_h} we see that μ_h^* is a zero-one measure.

Thus $F_{\mu_h} = \{t\}$ for some $t \in T$ and from the results of Sec. 2

$$h(f) = \int_{\{t\}} f \, d\mu_h^* = f(t).$$

We close the paper by noting that in a subsequent paper further characterizations of F — Q spaces analagous to those characterizations that exist for the classical notion [14, pp. 206, 207] will be considered. The purpose of developing these characterizations will be multi-fold. One is to produce examples of spaces which are not F — Q spaces. Another is to develop an analog of a result of Nachbin and Shirota [9, 11] which would characterize F–bornological function algebras. A third is to consider further structural properties of function algebras whose analysis is considerably helped in the classical setting by these characterizations. Other topologies such as the point-open topology on F(T) will also be investigated.

4. Addendum

It can be shown that if T is a O–dimensional Hausdorff space and F is any complete nonarchimedean nontrivially valued field, then a bounded continuous function taking a compact subset of T into F can be extended to a bounded continuous function taking T into F. As a result of this, all statements in Sec. 2 of this paper, in which the field is assumed to be discretely valued, are true for any complete nonarchimedean nontrivially valued field F.

Consequently we may prove that " F(T) is F–barreled if and only if for every subset E of T which is closed but not compact there is a function $f \in F(T)$ which is unbounded on E " assuming F to be any spherically complete field. This is again the unpublished result obtained by R. L. Ellis. However, by the methods of this paper, the support set of a continuous linear functional on F(T) is shown to be unique.

We also Note that in *Function Algebras over Valued Fields and Measures III* (to appear) we have shown that the collection Z(F(T)) of subsets of T is closed with respect to the operation of taking finite intersections as well as proved the result of Lemma 1 of Sec. 3 for any complete nonarchimedean nontrivially valued field F. Hence in all results of Sec. 3, " $\sqrt{-1} \notin F$ " may be removed. Of particular interest is the consequence that a Fréchet full algebra over any local field is functionally continuous.

Subsequent investigation has shown that the notion of " F — Q space " is independent of the field in the following sense: If T is an F — Q space for some complete discretely valued field F whose residue class field has nonmeasurable cardinal, then T is an F — Q space for all such fields F. This will appear in a subsequent paper.

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