Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **52** (1972), n.1, p. 73–79.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_52_1_73_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1972.

SEZIONE II

(Fisica, chimica, geologia, paleontologia e mineralogia)

Fisica teorica. — *On the* SO(3)×SO(3) *symmetry in hydrodynamical problems*. Nota di FRANCESCO PEGORARO, presentata (*) dal Corrisp. L. A. RADICATI DI BROZOLO.

RIASSUNTO. — Di alcuni problemi idrodinamici viene indicato un equivalente meccanico. Di questo vengono riconosciute le proprietà di invarianza e ad esse ricollegato un Teorema di Riemann. Si presenta inoltre una generalizzazione di questo Teorema.

I.

In this paper we examine the symmetry properties of closed, dissipation free hydrodynamical systems which are thermodynamically characterized by a relation between pressure and density.

The fact that the system is closed implies the constancy of the total angular momentum and therefore invariance under the SO(3) group. From the vorticity one can construct another operator which, under suitable conditions, forms with the angular momentum a basis of the SO(3)×SO(3) algebra. The suitable conditions referred to are:

(*i*) the existence of a linear relation between the Euler and Lagrange variables which is equivalent to the assumption that the position of a fluid element at time t is a linear function of the position at time t_0.

(*ii*) the validity of Dedekind's duality principle.

Following Dyson [1] we will use the correspondence between the class of hydrodynamical problems under consideration and a mechanical system. To describe the latter and to show its SO(3)×SO(3) invariance we will find it convenient to use a Lagrangian formulation. Another problem, already considered by Chandrasekhar [2], will be shown to possess the same symmetry.

We will then show that Riemann's Theorem [3] follows directly from the conservation of the SO(3)×SO(3) generators.

Finally we will prove a generalization of this Theorem by considering the consequences of the conservation of the generators under more general assumptions than those of Riemann's Theorem.

(*) Nella seduta del 15 gennaio 1972.

II.

We will consider a hydrodynamical system, closed and dissipation free, with a prescribed relation between pressure and density.

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be the Euler coordinate of a fluid element and $\boldsymbol{a} = (a_1, a_2, a_3)$ be its corresponding Lagrange coordinate.

We shall consider only solutions of the Euler equation that:

(*i*) are linear, i.e.

$$\boldsymbol{x} = \mathbf{F}(t)\,\boldsymbol{a} \tag{1}$$

where $\mathbf{F}$ is a non singular 3×3 matrix;

(*ii*) satisfy Dedekind's duality principle which states that if $\boldsymbol{x}$ is a solution also $\boldsymbol{x}' = {}^t\mathbf{F}(t)\,\boldsymbol{a}$ is a solution (1).

The matrix $\mathbf{F}$ can be canonically decomposed as

$$\mathbf{F} = \mathbf{T}\mathbf{D}\,{}^t\mathbf{S} \tag{2}$$

where $\mathbf{T}$ and $\mathbf{S}$ are orthogonal matrices and $\mathbf{D}$ is a diagonal one: $\mathbf{D}$, which is uniquely determined by $\mathbf{F}$, gives the shape of the density distribution of the fluid; $\mathbf{T}$ and $\mathbf{S}$ are defined modulo the intersection of the isotropy group of $\mathbf{D}$ with the orthogonal group; $\mathbf{S}$ and $\mathbf{T}$ specify respectively the orientation of the fluid with respect to the Lagrange and Euler coordinates.

(*iii*) We assume that $\mathbf{D}$ is positive definite.

Solutions satisfying (*i*) are called uniform motions; they are specified by the dilation velocity $\dot{\mathbf{D}}$, the angular velocity $\boldsymbol{\Omega} = \mathbf{T}\,{}^t\dot{\mathbf{T}}$ and the vorticity $\zeta = 2\,\boldsymbol{\Omega} + \mathbf{T}\zeta'\,{}^t\mathbf{T}$, where $\zeta' = -\mathbf{D}\boldsymbol{\Lambda}\mathbf{D}^{-1} - \mathbf{D}^{-1}\boldsymbol{\Lambda}\mathbf{D}$, with $\boldsymbol{\Lambda} = \mathbf{S}\,{}^t\dot{\mathbf{S}}$. All these quantities depend only upon t but are independent of $\boldsymbol{x}$.

Uniform motions of a fluid are completely determined by the knowledge of $\mathbf{F}$ as a function of time: the hydrodynamical problem can thus be reduced to a mechanical one with, in general, nine degrees of freedom. Under suitable conditions the latter can be described by a Lagrangian $\mathrm{L} = \mathrm{L}(\mathbf{F}, \dot{\mathbf{F}})$.

Because of (*ii*) $\mathrm{L}({}^t\mathbf{F}, {}^t\dot{\mathbf{F}}) = \mathrm{L}(\mathbf{F}, \dot{\mathbf{F}})$.

To analyze the symmetry of a system satisfying (*i*), (*ii*) and (*iii*) we begin by considering the special case discussed by Dyson [1] namely an isothermal non-interacting gas whose density is constant on ellipsoidal surfaces and is gaussian in the radial variable $\left(\dfrac{x_1^2}{d_1^2} + \dfrac{x_2^2}{d_2^2} + \dfrac{x_3^2}{d_3^2}\right)^{1/2}$ (2). It may be shown that the evolution of the corresponding mechanical system can be

(1) The superscript t denotes transposition.

(2) d_1 are the ellipsoid semiaxes.

derived from the Lagrangian

$$\mathrm{L}\,(\mathbf{F}\,,\dot{\mathbf{F}}) = \frac{1}{2}\,\mathrm{tr}\,(\dot{\mathbf{F}}\,{}^t\dot{\mathbf{F}}) - \mathrm{U} \tag{3}$$

where U, which is determined by the equation of state, depends upon det **F** and corresponds to the internal energy density of the hydrodynamical problem.

The canonical Poisson brackets are

$$\begin{aligned} \{\mathrm{F}_{ij}\,,\mathrm{F}_{kl}\} &= 0 \\ \{\dot{\mathrm{F}}_{ij}\,,\dot{\mathrm{F}}_{kl}\} &= 0 \\ \{\mathrm{F}_{ij}\,,\dot{\mathrm{F}}_{kl}\} &= -\,\delta_{ik}\,\delta_{jl} \end{aligned} \tag{4}$$

where F_{ij} and $\dot{\mathrm{F}}_{ij}$ are the matrix elements of **F** and $\dot{\mathbf{F}}$ and are respectively the coordinates and momenta of the mechanical system.

The Lagrangian (3) is invariant under SO(3)×SO(3) when its action on **F** is defined by

$$\mathbf{F} \rightarrow \mathbf{O}_1\,\mathbf{F}\,{}^t\mathbf{O}_2 \tag{5}$$

where $\mathbf{O}_1$ and $\mathbf{O}_2$ are orthogonal 3×3 matrices. The corresponding action on the Euler and Lagrange variables is

$$\boldsymbol{x} \rightarrow \mathbf{O}_1\,\boldsymbol{x} \qquad \boldsymbol{a} \rightarrow \mathbf{O}_2\,\boldsymbol{a}\,. \tag{6}$$

It then follows from Nöther's Theorem that the two screw symmetrical matrices

$$\begin{aligned} \mathbf{J} &= \mathbf{F}\,{}^t\dot{\mathbf{F}} - \dot{\mathbf{F}}\,{}^t\mathbf{F} \\ \mathbf{K} &= {}^t\mathbf{F}\dot{\mathbf{F}} - {}^t\dot{\mathbf{F}}\mathbf{F} = -\,{}^t\mathbf{F}\mathbf{J}\mathbf{F} \end{aligned} \tag{7}$$

are conserved. It is easy to see that **J** is proportional to the angular moment.

Using Eq. (4) one can readily prove that **J** and **K** satisfy the SO(3)×SO(3) algebra.

From this example one sees that (*i*) and (*ii*) are sufficient to guarantee SO(3)×SO(3) invariance. Indeed (*i*) ensures conservation of **J**. Under $\mathbf{F} \rightarrow {}^t\mathbf{F}$, **J** and **K** are interchanged and therefore **K** is also conserved.

Another interesting case satisfying (*i*), (*ii*) and (*iii*) but different from Dyson's is the one studied by Chandrasekhar [2], i.e. a homogeneous incompressible finite fluid which interacts gravitationally with itself.

The equation of state and the boundary conditions are:

α) the density does not depend on $\boldsymbol{x}$ and t;

β) the free surface is an ellipsoid;

γ) the pressure vanishes at the surface and is given by:

$p\,(\boldsymbol{x}) = p_c\left(1 - \sum_{i=1}^{3} x_i^2/d_i^2\right)$ where p_c is the pressure at the center of the fluid.

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The fluid is equivalent to an eight-dimensional mechanical system (notice that the volume is now fixed) and its evolution is derived from the Lagrangian

$$\mathrm{L}(\mathbf{F}, \dot{\mathbf{F}}) = \frac{1}{2}\,\mathrm{tr}\,(\dot{\mathbf{F}}\,{}^t\dot{\mathbf{F}}) - 2\pi\,\mathrm{G}\;\mathrm{tr}\,\mathbf{DAD} + \lambda\;\mathrm{tr}\log\mathbf{D} \tag{8}$$

where λ is a Lagrange multiplier arising from the condition det $\mathbf{F} = $ const; $\mathbf{A} = \mathbf{A}(\mathbf{D})$ is a diagonal matrix and G is the gravitational constant.

The kinetic term of the Lagrangian (8) is the same as in Dyson's problem. The second term arises from the gravitational interaction and is a function of $\mathbf{D}$. It is immediately verified that the second and third term in Eq. (8) are $SO(3) \times SO(3)$ invariant.

III.

As we have proved assumptions (*i*) and (*ii*) imply $SO(3) \times SO(3)$ invariance of the hydrodynamical problem. This invariance leads to the two equations:

$$\begin{aligned} 0 = \dot{\mathbf{J}} = \mathbf{T}\,(&-2\,\dot{\mathbf{D}}\boldsymbol{\Lambda}\mathbf{D} - 2\,\mathbf{D}\boldsymbol{\Lambda}\dot{\mathbf{D}} + 2\,\mathbf{D}\dot{\mathbf{D}}\boldsymbol{\Omega} + 2\,\boldsymbol{\Omega}\mathbf{D}\dot{\mathbf{D}} + 2\,\boldsymbol{\Omega}\mathbf{D}\boldsymbol{\Lambda}\mathbf{D} + \\ &- 2\,\mathbf{D}\boldsymbol{\Lambda}\mathbf{D}\boldsymbol{\Omega} + \mathbf{D}^2\boldsymbol{\Omega}^2 - \boldsymbol{\Omega}^2\mathbf{D}^2 + \mathbf{D}^2\dot{\boldsymbol{\Omega}} + \dot{\boldsymbol{\Omega}}\mathbf{D}^2 - 2\,\mathbf{D}\dot{\boldsymbol{\Lambda}}\mathbf{D})\,{}^t\mathbf{T} \\ 0 = \dot{\mathbf{K}} = \mathbf{S}\,(&-2\,\dot{\mathbf{D}}\boldsymbol{\Omega}\mathbf{D} - 2\,\mathbf{D}\boldsymbol{\Omega}\dot{\mathbf{D}} + 2\,\mathbf{D}\dot{\mathbf{D}}\boldsymbol{\Lambda} + 2\,\boldsymbol{\Lambda}\mathbf{D}\dot{\mathbf{D}} + 2\,\boldsymbol{\Lambda}\mathbf{D}\boldsymbol{\Omega}\mathbf{D} + \\ &- 2\,\mathbf{D}\boldsymbol{\Omega}\mathbf{D}\boldsymbol{\Lambda} + \mathbf{D}^2\boldsymbol{\Lambda}^2 - \boldsymbol{\Lambda}^2\mathbf{D}^2 + \mathbf{D}^2\dot{\boldsymbol{\Lambda}} + \dot{\boldsymbol{\Lambda}}\mathbf{D}^2 - 2\,\mathbf{D}\dot{\boldsymbol{\Omega}}\mathbf{D})\,{}^t\mathbf{S} \end{aligned} \tag{9}$$

where $\boldsymbol{\Omega} = \mathbf{T}\,{}^t\dot{\mathbf{T}}$ and $\boldsymbol{\Lambda} = \mathbf{S}\,{}^t\dot{\mathbf{S}}$.

Let us consider the case

$$\dot{\mathbf{D}} = \dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\Lambda}} = 0\,.$$

From Eq. (9) we get

$$\begin{aligned} 0 &= 2\,\boldsymbol{\Omega}\mathbf{D}\boldsymbol{\Lambda}\mathbf{D} - 2\,\mathbf{D}\boldsymbol{\Lambda}\mathbf{D}\boldsymbol{\Omega} + \mathbf{D}^2\boldsymbol{\Omega}^2 - \boldsymbol{\Omega}^2\mathbf{D}^2 \\ 0 &= 2\,\boldsymbol{\Lambda}\mathbf{D}\boldsymbol{\Omega}\mathbf{D} - 2\,\mathbf{D}\boldsymbol{\Omega}\mathbf{D}\boldsymbol{\Lambda} + \mathbf{D}^2\boldsymbol{\Lambda}^2 - \boldsymbol{\Lambda}^2\mathbf{D}^2 \end{aligned} \tag{10}$$

We want now to show the equivalence of these equations to Riemann's Theorem: if a hydrodynamical system satisfies (*i*), (*ii*) and (*iii*) and $0 = \dot{\mathbf{D}} = \dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\Lambda}} \Longleftrightarrow \dot{\zeta}' = 0$ then either $\boldsymbol{\Omega}$ is parallel to ζ' and they both lie on a principal axis, or it is not parallel to ζ' and the plane they form is a principal one.

Riemann's Theorem is a necessary condition only as it does not involve the interaction.

Chandrasekhar has given a proof of Riemann's Theorem for the system described in [2] using the nondiagonal elements of the second order virial.

To prove the equivalence of equations (10) to Riemann's Theorem let us write them for the (2, 3)–component:

$$\begin{aligned}2\,d_1\,d_3\,\Omega_{12}\,\Lambda_{13}-2\,d_1\,d_2\,\Omega_{13}\,\Lambda_{12}&=(d_3^2-d_2^2)\,\Omega_{12}\,\Omega_{13}\\ 2\,d_1\,d_3\,\Lambda_{12}\,\Omega_{13}-2\,d_1\,d_2\,\Lambda_{13}\,\Omega_{12}&=(d_3^2-d_2^2)\,\Lambda_{12}\,\Lambda_{13}\,.\end{aligned}\tag{11}$$

Let us suppose that the components of $\boldsymbol{\Omega}$, Ω_{12} and Ω_{13} say, are different from zero. Using the relation $\zeta'_{ik}=-\dfrac{d_i^2+d_k^2}{d_i\,d_k}\Lambda_{ik}$ we get after some algebraic manipulations:

$$d_2^2+d_3^2+\frac{2\,d_1^2\,d_3^2}{d_1^2+d_3^2}\frac{\zeta'_{13}}{\Omega_{13}}+\frac{2\,d_1^2\,d_2^2}{d_1^2+d_2^2}\frac{\zeta'_{12}}{\Omega_{12}}+ \tag{12}$$

$$+\frac{2\,d_1^2\,d_2^2\,d_3^2}{(d_1^2+d_3^2)\,(d_1^2+d_2^2)}\frac{\zeta'_{13}}{\Omega_{13}}\frac{\zeta'_{12}}{\Omega_{12}}=0$$

$$d_3^2+\frac{2\,d_1^2\,d_3^2}{d_1^2+d_3^2}\frac{\zeta'_{13}}{\Omega_{13}}=d_2^2+\frac{2\,d_1^2\,d_2^2}{d_1^2+d_2^2}\frac{\zeta'_{12}}{\Omega_{12}}\,.$$

Solving (12) for ζ'_{13}/Ω_{13} we get:

$$\left(\frac{\zeta'_{13}}{\Omega_{13}}\right)^2+(4\,d_1^2-d_2^2+d_3^2)\,\frac{d_1^2+d_3^2}{2\,d_1^2\,d_3^2}\left(\frac{\zeta'_{13}}{\Omega_{13}}\right)+\frac{(d_1^2+d_3^2)^2}{d_1^2\,d_3^2}=0. \tag{13}$$

If also Ω_{23} were different from zero, we would obtain with a similar procedure $d_1=d_2=d_3$.

Hence, excluding the trivial case, $\mathbf{D}$ proportional to the identity, only two pairs $(\zeta'_{ik},\Omega_{ik})$ can be different from zero: this is an equivalent formulation of Riemann's Theorem.

Using Eq. (9) we can discuss cases which are more general than the one considered in Riemann's Theorem. For example let us suppose that only $\dot{\mathbf{D}}$ vanishes. Equations (9) are a system of six equations for the fifteen quantities $\boldsymbol{\Omega}$, $\dot{\boldsymbol{\Omega}}$, $\boldsymbol{\Lambda}$, $\dot{\boldsymbol{\Lambda}}$, $\mathbf{D}$.

We examine in particular the case where:

$$\dot{\boldsymbol{\Lambda}}(t)=f[\boldsymbol{\Lambda}(t)] \quad \text{and} \quad \dot{\boldsymbol{\Omega}}(t)=g[\boldsymbol{\Omega}(t)] \tag{14}$$

and we restrict our attention to the special subcase when $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}$ preceed around a common axis, the third say. This means:

$$\begin{aligned}\Omega_{12}&=\text{const} & \Lambda_{12}&=\text{const}\\ \Omega_{13}&=\Omega_0\cos(\omega_1 t) & \Lambda_{13}&=\Lambda_0\cos(\omega_2 t+\varphi)\\ \Omega_{23}&=\Omega_0\sin(\omega_1 t) & \Lambda_{23}&=\Lambda_0\sin(\omega_2 t+\varphi) \qquad \Omega_0\,,\ \Lambda_0>0\,.\end{aligned}\tag{15}$$

We shall prove that as a consequence of the SO(3)×SO(3) symmetry:

1) the precession velocities ω_1 and ω_2 are equal and the phase difference φ is zero;

2) the ellipsoid reduces to a spheroid with axial symmetry around the precession axis;

3) the quantities $d_1\, d_3\, \Omega_{12}\, \Lambda_{12}\, \Omega_0\, \Lambda_0\, \omega = \omega_1 = \omega_2$ satisfy Eq. (22).

To prove this let us consider the component of Eq. (9) on the rotation axis (1 , 2)

$$(16) \quad \begin{aligned} 0 &= 2\,(d_2\, d_3\, \Omega_0\, \Lambda_0 \cos(\omega_1 t) \sin(\omega_2 t + \varphi) - d_1\, d_3\, \Omega_0\, \Lambda_0 \sin(\omega_1 t) \cdot \\ &\quad \cdot \cos(\omega_2 t + \varphi) + (d_2^2 - d_1^2)\, \Omega_0^2 \cos(\omega_1 t) \sin(\omega_1 t)) \\ 0 &= 2\,(d_2\, d_3\, \Omega_0\, \Lambda_0 \sin(\omega_1 t) \cos(\omega_2 t + \varphi) - d_1\, d_3\, \Omega_0\, \Lambda_0 \cos(\omega_1 t) \cdot \\ &\quad \cdot \sin(\omega_2 t + \varphi) + (d_2^2 - d_1^2)\, \Lambda_0^2 \cos(\omega_2 t + \varphi) \sin(\omega_2 t + \varphi)). \end{aligned}$$

Since these equations must hold every t, the coefficients of the terms with different frequencies must vanish independently. The following cases can occur:

(*i*) The two eigenvalues of $\mathbf{D}, d_1$ and d_2, are different. This implies $\omega_1 = \pm\, \omega_2$.

(*i'*) If $\omega_1 = \omega_2 = \omega$ then

$$(17) \quad \begin{aligned} 0 &= d_2\, d_3\, \Omega_0\, \Lambda_0 [\sin(2\,\omega t + \varphi) + \sin\varphi] - d_1\, d_3\, \Omega_0\, \Lambda_0 \cdot \\ &\quad \cdot [\sin(2\,\omega t + \varphi) - \sin\varphi] + \frac{1}{2}\,(d_2^2 - d_1^2)\, \Omega_0^2 \sin(2\,\omega t + 2\,\varphi) \end{aligned}$$

which implies $\varphi = 0$.

However from (17) and the second equation (16) we have $\Lambda_0 = \Omega_0$ and therefore

$$(18) \quad 2\, d_3 = -(d_2 + d_1)$$

which is incompatible with the positive definite character of $\mathbf{D}$.

(*i''*) If $\omega_1 = -\omega_2 = \omega$ with a similar procedure we get

$$(19) \quad \varphi = 0 \quad \Lambda_0 = \Omega_0 \quad , \quad 2\, d_3 = d_1 - d_2$$

(*ii*) The two eigenvalues d_1 and d_2 are equal. This implies

$$\omega_1 = \omega_2 = \omega\,.$$

As before $\varphi = 0$ but $d_1, d_3, \Omega_0, \Lambda_0$ remain arbitrary.

With a similar argument we can derive for the subcase (*i''*) from the components of Eq. (9) on the plane orthogonal to the rotation axis the two conditions:

$$(21) \quad \begin{aligned} \Omega_{12}\,(d_1 - 3\, d_3) + (d_3 + d_1)\, \omega &= 0 \\ \Omega_{12}\,(d_2 - 3\, d_3) + (d_3 + d_2)\, \omega &= 0\,. \end{aligned}$$

It is easy to see that (21) and (19) lead to $d_2 = -d_3$ which is absurd. In this way we have proved that case (*i*) is excluded.

With a similar technique we can show that case (*ii*) leads to no contradiction provided that the variables satisfy the equations

$$0 = 2\,d_1\,(d_3\,\Omega_{12}\,\Lambda_0 - d_1\,\Lambda_{12}\,\Omega_0) + (d_3^2 - d_1^2)\,\Omega_{12}\,\Omega_0 + \\ + (d_1^2 + d_3^2)\,\omega\Omega_0 - 2\,d_1\,d_3\,\omega\Lambda_0$$

$$0 = 2\,d_1\,(d_3\,\Lambda_{12}\,\Omega_0 - d_1\,\Omega_{12}\,\Lambda_0) + (d_3^2 - d_1^2)\,\Lambda_{12}\,\Lambda_0 + \\ + (d_1^2 + d_3^2)\,\omega\Lambda_0 - 2\,d_1\,d_3\,\omega\Omega_0 \qquad \text{Q.E.D.}$$

Aknowledgements. – I would like to thank Prof. L. A. Radicati for several discussions that helped me to clarify the problem and write the paper.

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